EULER-POINCARÉ OPERATORS FOR ARCHITECTURAL COMPLEXES

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ABSTRACT. We define the notion of *architectural complex* as a weighted graph. From a database point of view, architectural complexes are a straightforward way of encoding chain complexes from volume modelling, as effected in [2]. We show that the notion of *Euler operator* from volume modelling generalises to architectural complexes. In arbitrary dimension, we call them *Euler-Poincaré operators*, as they fulfill the Euler-Poincaré equation of homology. We introduce *elementary* Euler-Poincaré operators for architectural complexes and show that there are finitely many, and they generate all Euler-Poincaré operators. Finally, we algorithmically construct cellulations from topological realisations of architectural complexes using Euler-Poincaré operators in order to compute their topological Betti numbers.

1. INTRODUCTION

Euler operators were introduced into volume modelling by Baumgart [1] as transactions which modify a volume model while at the same time maintaining the Euler relation

$$F - E + V = 2B - 2H$$

between the numbers of faces (F), edges (E), vertices (V), bodies (B) and handles (H) of polyhedral surfaces. These operators are an important tool in boundary representation modeling [7, 10]. By encoding the cellular decomposition of a spatial object in a chain complex $\mathcal{C}$, the Euler relation translates into the consistency rule

$$\partial \circ \partial = 0$$

for the boundary operator ∂ of the chain complex. This chain complex $\mathcal{C}$ is an algebraic description of the cellular decomposition, and the consistency rule means topologically that the boundary itself has no boundary, i.e. is a cycle. For example, in Figure 1 the boundary of the disk as a cw-complex is the graph forming a loop. Another example: the boundary of a ball is a sphere, and its boundary is empty space. Through the advent of algebraic topology, the study of chain complexes

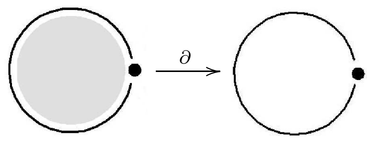

FIGURE 1. Boundary is a cycle.

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became of independent interest. Poincaré generalised the Euler relation to chain complexes of arbitrary finite dimension to the Euler-Poincaré formula

$$\sum_{i=0}^{n}(-1)^i v_i = \sum_{i=0}^{n}(-1)^i b_i$$

where v_i is the number of i-cells and b_i the i-th Betti number of the complex. The latter numbers have, in the case of complexes derived from cellular decompositions, the topological interpretation as numbers of connected components, loops or higher dimensional "holes". The boundary of a chain complex being representable by a matrix makes that notion interesting as a data structure for spatial information [2, 8].

In this article, we consider chain complexes which allow a topological interpretation as decompositions of a space into more general blocks which are not necessarily cells. These blocks we call *regions*, and the data structure derived therefrom *architectural complex*. E.g. a sphere can appear as a constituent which is not a cell. This forces the Betti numbers not to be directly interpreted topologically, but we will show that by transforming architectural complexes into cw-complexes the defects can be computationally controlled. This is realised by a higher-dimensional form of Euler operators which we hence call *Euler-Poincaré operators*. Our motivation to consider this degree of generality comes in fact from architecture, where building elements often have "holes" and decompositions into cells or simplices seems unnatural from the architectural viewpoint. An important feature of the notion "architectural complex" is that it uses partial matrices for representing the boundary operator instead of full matrices as in chain complexes. This has the advantage of reducing the data size and regaining some of the topolgical information otherwise lost in the chain complexes [2]. Viewing partial matrices as a special case of a set with a relation, one obtains a modification of the relational database model which encaptures topological information [9].

Extensions of the notion of Euler-operator to more general cases can also be found in [3, 5, 6].

In this article we assume some familiarity with basic notions from set-theoretic and algebraic topology, in particular the notions *cw-complex*, *chain complex* as well as their homology groups. These prerequisites can be obtained in a condensed form by reading [2, Sections 2–4]. As a more in-depth introduction to the relevant notions of algebraic topology can serve the book [4].

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2. Euler-Poincaré formula

Let

$$\mathcal{C}\colon C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

be a chain complex of finitely generated free $\mathbb{Z}$-modules C_i with fixed basis B_i each, and boundary operator $\partial\colon C \to C$ such that $\partial|_{C_i} = \partial_i$. The rank of each module C_i is finite and equals $v_i = \#B_i$. The homology groups

$$H_i(\mathcal{C}, \mathbb{Z}) = \ker \partial_i / \operatorname{im} \partial_{i+1}$$

are also finitely generated. The rank of the torsion-free part of $H_i(\mathcal{C}, \mathbb{Z})$ is denoted by $b_i = \operatorname{rk} H_i(\mathcal{C}, \mathbb{Z})$ and is known as the *i-th Betti number* of $\mathcal{C}$.

If $\mathcal{C}$ is the chain complex associated to a cw-complex, an element of B_i is called an *i-cell*. This denomination comes from the fact, that in a cw-complex C is built up from i-dimensional cells (i.e. "deformed" open i-dimensional Euclidean balls), and these form a basis B_i for the chain module C_i of the associated chain complex $\mathcal{C}$. Hence, if the chain complex $\mathcal{C}$ is associated to a cw-complex C, then V_i is in fact the number of i-cells of C and b_i is the number of i-dimensional "holes" in C.

The following theorem is well known and can be found e.g. in [4, Theorem 2.44].

Theorem 2.1 (Euler-Poincaré formula)**.**

$$\sum_{i=0}^{n}(-1)^i v_i = \sum_{i=0}^{n}(-1)^i b_i$$

For the convenience of the reader, we sketch here a proof of this standard fact.

Proof. By definition of $H_0(\mathcal{C}, \mathbb{Z}) = C_0/\operatorname{im}\partial_1$, it follows that

$$v_0 = b_0 + j_1,$$

where j_ℓ is the rank of the $\mathbb{Z}$-module $\operatorname{im}\partial_\ell$ for $\ell = 1, \dots, n$. And by linear algebra, it holds true that

$$v_\ell = k_\ell + j_\ell, \tag{1}$$

where k_ℓ is the rank of $\ker\partial_\ell$. Hence, it follows that

$$v_0 = b_0 + V_1 - k_1. \tag{2}$$

Again, by definition of the first homology group, it holds true that

$$k_1 = b_1 + j_2.$$

Using this and (1), equation (2) becomes

$$v_0 = b_0 + v_1 - b_1 - j_2 = b_0 + v_1 - b_1 - v_2 + k_2.$$

Induction and the fact that $b_n = k_n$ completes the proof. □

3. Architectural complexes

Chain complexes are often derived from cellular decompositions of manifolds, or more general topological spaces. They contain the algebraic realisations of cw-complexes and provide a convenient way of constructing data structures for spatial objects which can be for example used in the foundation of building information models [9, 8, 2]. However, its use is somewhat restricted due to information losses on the way from cw-complexes to chain complexes (already in the case of graphs) [2]. The reason is that the topological interpretation of a zero entry in the matrix for the boundary operator is not clear. This is apparent already in the case of graphs, as illustrated in Figure 2, where two possible types of graphs have the same chain complex (the vertex adherent to the loop is not clear).

$$\mathcal{C}: \quad \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c \xrightarrow{\partial} \mathbb{Z}P \oplus \mathbb{Z}Q \oplus \mathbb{Z}R,$$
$$a \mapsto Q - P,\ b \mapsto R - Q,\ c \mapsto 0$$

It was found that defining a matrix entry $\delta(b, c)$ only in the case of adherence between the cells b and c (b lies in the closure of c) and leaving the matrix entry

FIGURE 2. Two non-isomorphic graphs with isomorphic chain complex.

undefined otherwise, recovers at least partially the information loss [2]. Table 1 gives the matrix and partial matrices for the graphs in Figure 2 (δ for the left, δ' for the right graph). In both partial matrices precisely one zero entry is kept in two different topologically relevant places, whereas all other zero entries are discarded. At the same time the advantage of having the simple data structure given by a

∂	a	b	c
P	-1	0	0
Q	1	-1	0
R	0	1	0

δ	a	b	c
P	-1		
Q	1	-1	0
R		1	

δ'	a	b	c
P	-1		
Q	1	-1	1
R		1	0

TABLE 1. Tables representing boundary and partial boundary operators.

(now partial) matrix is retained. Even more, the boundary operator being a partial matrix reduces the amount of stored zeros significantly. As the idea originated in architecture, these new objects will be named *architectural complexes*.

3.1. **Architectural complexes and weighted graphs.** The consideration of architectural complexes uses some notions of partial linear algebra which will be introduced along the way.

Definition 3.1. *An* architectural complex $\mathscr{C}$ *is given by the following data, resp. conditions:*

(1) *a finite set B partitioned into subsets: $B = \bigcup_{i=0}^{n} B_i$,*
(2) *a partial matrix $\delta :\subseteq B \times B \to \mathbb{Z}$ which is defined on $x \in B_j \times B_i$ only if $j < i$, and $\delta(b, c) = 0$ if the expression is defined with $b \in B_j$, $c \in B_i$ for $j < i - 1$.*
(3) *The relation $\delta^2 = 0$ holds true for some partial matrix $0 :\subseteq B \times B \to \mathbb{Z}$ whose entries are all zero or undefined.*

The elements of B_i are called i-regions, *and the partial matrix δ is called the* partial boundary operator *of $\mathscr{C}$. By a* region *we mean any $b \in B$. An architectural complex will also be denoted as*

$$\mathscr{C} :\subseteq C_n \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_1} C_0.$$

Note that the multiplication of partial matrices is defined in the same way as the multiplication of usual matrices, whereby using the rules

$$a + \text{undefined} = a$$
$$a \cdot \text{undefined} = \text{undefined}$$

In particular, we use the rule $0 \cdot \text{undefined} = \text{undefined}$.

An architectural complex $\mathscr{C}$ can be viewed as a weighted directed graph Γ with edges determined by the maximal subset of $B \times B$ on which δ is defined, and the weight of an edge $e = (b, c)$ is given by the value of δ in e. The interpretation of condition (2) is that the boundary of a region c contains only regions b of lower dimension, and if the dimension jump is more than one, then all "sides" of b are adjacent to c. The last condition (3) means that the square of the adjacency matrix δ yields a graph whose edge weights are all zero. We will say that $\Gamma = \Gamma(\mathscr{C})$ is the *underlying graph* of $\mathscr{C}$. Often, this graph is called *incidence graph.*

Lemma 3.2. *The underlying graph $\Gamma(\mathscr{C})$ of an architectural complex $\mathscr{C}$ is a (weighted) directed acyclic graph.*

Proof. This is clear from the definition. □

We will depict the graph $\Gamma(\mathscr{C})$ of an architectural complex $\mathscr{C}$ by drawing a vertex decorated with a weight n for each n-region. An edge (b, c) is drawn as in Figure 3 by an arrow from the higher-dimensional region c to the region b of lower dimension. This will be the orientation chosen for all edges in $\Gamma(\mathscr{C})$.

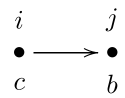


Figure 3. Graphical representaion of an architectural complex.

Remark 3.3. *Assume now the notation as in Section 2. If $\mathcal{C}$ is the chain complex associated to a cw-complex X, then by removing all topologically irrelevant zeros from the matrix representing ∂ with respect to the basis B yields an architectural complex $\mathscr{C}(X)$* associated to X. *Later we will, by abuse of language, often not distinguish between a cw-complex and its associated architectural complex.*

Example 3.4. *Figure 4 illustrates some architectural complexes which are not cw-complexes. The reason is in the first and third case that the lines have no boundary. In the second and third case, there is another reason: namely the boundary of the surface is not connected. The weight ± 1 means that it is either $+1$ or -1, depending on the orientation imposed on the regions.*

3.2. **Topology of architectural complexes.** Let $\mathscr{C}$ be an architectural complex. The underlying graph $\Gamma(\mathscr{C})$ defines a topology on the set B of regions which we describe in the following.

Let $b \in B$. The set

$$\mathrm{Star}(b) := \{c \in B \mid \exists \text{ a path } c \rightsquigarrow b \text{ in } \Gamma(\mathscr{C})\}$$

is called the *star* of b in $\mathscr{C}$.

Definition 3.5. *The topology $\mathcal{T}$ generated by all stars in $\mathscr{C}$ is called the* underlying topology *of $\mathscr{C}$. The topological space $(B, \mathcal{T})$ will often be denoted by $|\mathscr{C}|$ and is called the* underlying topological space *of the architectural complex $\mathscr{C}$.*

The underlying topology of an architectural complex is a special case of the more general notion of a topology defined by a relation on set [2]. The underlying

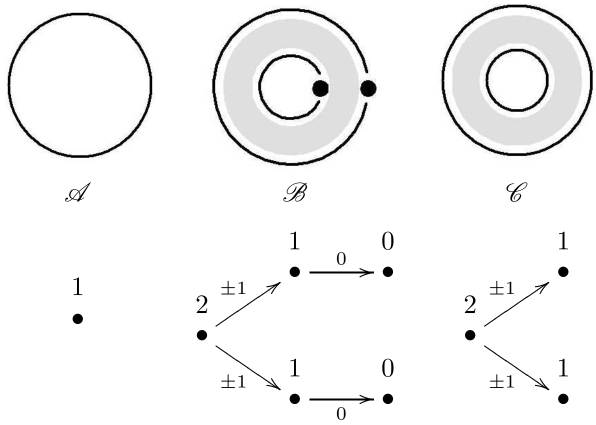


FIGURE 4. Architecural complexes which are not cw.

topology of an architectural complex is obviously not Hausdorff. A well-known and important result which, however, will not be used in the remainder of this article, is that it satisfies the T_0-axiom:

Axiom 3.6. *For any two distinct points $x, y \in |\mathscr{C}|$, there is an open set U such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.*

Proof. Namely, if $x \notin \mathrm{Star}(y)$ then take $U = \mathrm{Star}(y)$, otherwise take $U = \mathrm{Star}(x)$. The case that $\mathrm{Star}(x) = \mathrm{Star}(y)$ cannot happen, because $\Gamma(\mathscr{C})$ is acyclic. □

Remark 3.7. *If $\mathscr{C}$ comes from a cw-complex X, then the underlying topological spaces $|\mathscr{C}|$ and $|X|$ differ, where $|X|$ bears the weak topology. However, in* [2], *cw-complexes are viewed as (finite) combinatorial objects, of which architectural complexes are a natural generalisation. In this case, the topological space is precisely the underlying topological space of $\mathscr{C}$. Such architectural complexes are called* combinatorial cw-complexes *in* [2].

3.3. **Homology of architectural complexes.** In usual linear algebra, a matrix defines a linear map between free modules (or vector spaces) by giving bases for the source and target modules. In a similar manner, we will say that a partial matrix defines a partial linear map between free modules. Namely, let M and N be free modules with fixed bases B and C, respectively. Then a partial map $f :\subseteq B \to C$ is extended to a partial linear map $\Phi :\subseteq M \to N$ in the usual way by defining for any $x = \sum\limits_{b \in B} \alpha_b b \in M$

$$\Phi\left(\sum_{b\in B} \alpha_b b\right) := \sum_{b \in B} \alpha_b f(b)$$

and by applying in the case of $f(b) =$ undefined the rules from Section 3.1. However, the partial linear map defined in this way depends on the choices of bases for the source and target modules. This is not a serious problem in our case, because in our situation any free module is defined by fixing a basis.

Definition 3.8. *Let $f :\subseteq M \to N$ be a partial linear map given by a partial matrix φ with respect to some fixed bases B, C of the free modules M and N. Then the* *matrix*

$$\Phi\colon B \times C \to \mathbb{Z},\ (a,b) \mapsto \begin{cases} \varphi(a,b), & \textit{if expression is defined} \\ 0, & \textit{otherwise} \end{cases}$$

defines a linear map $F\colon M \to N$*, called the* completion of f by zero*. The* kernel *of* F *is defined as* $\ker F := \ker f$.

Let us remark that $\operatorname{im} f = \operatorname{im} F$ holds true for a partial linear map f and its completion by zero F. This allows us to define the homology groups of architectural complexes.

Definition 3.9. *Let* $\mathscr{C}$ *be an architectural complex with partial boundary operator* δ*. Then*

$$H_i(\mathscr{C}, \mathbb{Z}) := \ker \delta_i / \operatorname{im} \delta_{i+1}$$

is the i-th homology group *of* $\mathscr{C}$*.*

If $\mathscr{C} :\subseteq C_n \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_1} C_0$ is an architectural complex, then its partial boundary operator δ can be completed by zero to a linear map ∂. It is clear, that $\partial^2 = 0$ (complete zero matrix) holds true. Hence, ∂ is the boundary operator of a chain complex $\mathcal{C}$ whose underlying chain modules coincide with those of $\mathscr{C}$:

$$\mathcal{C}\colon C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} C_0$$

The chain complex $\mathcal{C}$ is called the *chain complex associated to* $\mathscr{C}$*.*

Lemma 3.10. *Let* $\mathscr{C}$ *be an architectural complex, and let* $\mathcal{C}$ *be the chain complex associated to* $\mathscr{C}$*. Then it holds true that*

$$H_i(\mathscr{C}, \mathbb{Z}) = H_i(\mathcal{C}, \mathbb{Z}).$$

Proof. This is obvious from the definitions. □

A trivial, but important, consequence is that the Euler-Poincaré formula is valid also for architectural complexes.

Corollary 3.11. *Let* $\mathscr{C}$ *be an architectural complex. Then*

$$\sum_{i=0}^{n} (-1)^i v_i = \sum_{i=0}^{n} (-1)^i b_i,$$

where $b_i = \operatorname{rk} H_i(\mathscr{C}, \mathbb{Z})$ *and* $v_i = \# B_i$ *with* B_i *as in Definition 3.1.*

Remark 3.12. *The interpretation of Betti numbers of general architectural complexes needs some care. E.g. there exist architectural complexes with* $b_0 = 0$*, or connected architectural complexes with* $b_0 > 1$*. Explicit examples are given in Example 3.13*

Example 3.13. *The Betti numbers of the architectural complexes in Figure 4 are as follows:*

$\mathscr{A}$	$\mathscr{B}$	$\mathscr{C}$
$b_0 = 0$	$b_0 = 2$	$b_0 = 0$
$b_1 = 1$	$b_1 = 1$	$b_1 = 1$
	$b_2 = 0$	$b_2 = 0$

This follows from the following considerations:

In $\mathscr{A}$ and in $\mathscr{C}$ there are no 0*-cells. Hence,* $b_0 = 0$ *in those cases. In $\mathscr{B}$, there are* 2 *vertices. This means that* $\ker \partial_0$ *is of rank* 2*. Since both lines have boundary zero, it follows that* $b_0(\mathscr{B}) = 2$*. The first Betti number can be calculated in this way:* $b_1(\mathscr{A})$ *is clearly the rank of* $\ker \partial_1$*, hence equals* 1*. In $\mathscr{B}$ and $\mathscr{C}$ both lines map to zero, and the boundary of the surface is the sum (or difference*[1]*) of the two lines. Hence* $b_1 = 2 - 1 = 1$ *in these cases. The result for* b_2 *follows from the fact that the surface has non-zero boundary in $\mathscr{B}$ and $\mathscr{C}$.*

4. Geometry of Euler-Poincaré space

The Euler-Poincaré formula (Theorem 2.1, or Corollary 3.11 if you like) is a linear constraint for the possible numbers v_i of i-cells and Betti numbers b_i a chain complex or architectural complex can have. So, we define

$$\mathfrak{E} := \{(V_0, \dots, V_n; b_0, \dots, b_n) \in \mathbb{Z}^{n+1} \times \mathbb{Z}^{n+1} \mid \sum_{i=0}^{n} (-1)^i (V_i - b_i) = 0\}.$$

The space $\mathfrak{E}$ is a hyperplane in $V := \mathbb{Z}^{n+1} \times \mathbb{Z}^{n+1}$ consisting of those points satisfying the Euler-Poincaré formula. Because cardinalities and Betti numbers cannot be negative, we consider the subspace $\mathbb{E} \subseteq \mathfrak{E}$ with non-negative coordinates.

Definition 4.1. *The set* $\mathfrak{E}$ *is called the* Euler-Poincaré hyperplane*, and*

$$\mathbb{E} := \mathfrak{E} \cap (\mathbb{N}^{n+1} \times \mathbb{N}^{n+1})$$

the Euler-Poincaré space.

The Euler-Poincaré hyperplane $\mathfrak{E}$ is a $2n + 1$-dimensional linear subspace of V, because it is the solution space of one linear equation. Hence, the sum and the difference of two points from $\mathfrak{E}$ also lies in $\mathfrak{E}$, and $\mathfrak{E}$ contains the zero vector $0 = (0, \dots, 0; 0, \dots, 0) \in V$. The geometry of $\mathfrak{E}$ turns out useful in the study of operations on architectural complexes through so-called Euler-Poincaré operators which act on $\mathbb{E}$, as will be seen in Section 5.

A basis for $\mathfrak{E}$, considered as the solution space of the equation

$$\sum_{i=0}^{n} (-1)^i (x_i - y_i) = 0, \tag{3}$$

is readily seen to be

$$K_i = X_i - (-1)^i X_0, \qquad i = 1, \dots, n \tag{4}$$

$$L_j = Y_j + (-1)^j X_0, \qquad j = 0, \dots, n \tag{5}$$

where $X_i = (e_i; 0), Y_j = (0; e_j) \in V$ and

$$e_\ell = (0, \dots, 0, \underbrace{1}_{\text{at position } \ell}, 0, \dots, 0), \qquad \ell = 0, \dots, n,$$

are the standard basis vectors in $\mathbb{Z}^{n+1}$.

[1]depending on the choice of orientations

The topological interpretation of the standard vectors X_i and Y_j and their negatives is:

$$\begin{aligned} X_i &: \text{“make” an } i\text{-dimensional “element”} \\ Y_j &: \text{“make” a } j\text{-dimensional “hole”} \\ -X_i &: \text{“kill” an } i\text{-dimensional “element”} \\ -Y_j &: \text{“kill” a } j\text{-dimensional “hole”} \end{aligned}$$

However, the meaning of Y_0 is usually “join” two “parts”.

Proposition 4.2. *A point $x \in \mathfrak{E}$ lies in $\mathbb{E}$ if and only if x is of the form*

$$x = \sum_{i=1}^{n} \alpha_i K_i + \sum_{j=0}^{n} \beta_j L_j \tag{6}$$

with $\alpha_i, \beta_j \in \mathbb{N}$ satisfying the inequality

$$\sum_{j=0} (-1)^j \beta_j - \sum_{i=0}^{n} (-1)^i \alpha_i \geq 0. \tag{7}$$

Proof. If $x \in \mathbb{E}$, then clearly, x is a linear combination (6) of K_i and L_j with non-negative coefficients α_i, β_j.

On the other hand, if x is expressed in terms of X_i and Y_j, then one observes with some straightforward calculation that the inequality (7) ensures the coefficient of X_0 to be non-negative. Since in (6) $\alpha_i, \beta_j \in \mathbb{N}$ is assumed, it is clear that the coefficients of the remaining X_i and the Y_j $(i, j = 1, \ldots, n)$ are all non-negative integers. Hence, $x \in \mathbb{E}$. □

5. Euler-Poincaré operators

In the previous section, we have introduced the Euler-Poincaré space $\mathbb{E}$ as the space of all possible configurations $(V_0, \ldots, V_n; b_0, \ldots, b_n)$ of cell numbers and Betti numbers which can occur in architectural complexes. The result of Proposition 4.2 is that the special configurations K_i, L_j allow, although themselves not contained in $\mathbb{E}$, to obtain all configurations lying in $\mathbb{E}$ by superposition.

A different point of view includes all of $\mathfrak{E}$. Namely, an element x of $\mathfrak{E}$ can be interpreted as the effecting of a change to some given complex c with configuration $e(c) \in \mathbb{E}$ by simply adding:

$$e(c') := e(c) + x \in \mathfrak{E}$$

is then the configuration for another complex c' obtained by changing the numbers of cells and Betti numbers according to x. This is the case if and only if $e(c') \in \mathbb{E}$. In this viewpoint, we call $x \in \mathfrak{E}$ an *Euler-Poincaré operator*.

Consider, for example, $K_1 = X_1 - X_0$ (make edge, kill vertex: mekv) and $L_0 = Y_1 - X_0$ (make loop, kill vertex: mlkv). The Euler-Poincaré operator K_1 introduces a new 1-cell and removes a 0-cell of any given complex on which it acts. And L_0 makes a 1-dimensional loop (i.e. raises b_1 by one) while at the same time removing a 0-cell. The like happens for the other K_i and L_j. The effect of K_i can be made undone by the Euler-Poincaré operator $-K_i$, and $-L_j$ yields the data started with before the change made by L_j. Hence, the set $\mathfrak{E}$ of Euler-Poincaré operators form an Abelian group generated by the K_i and L_j, where the group operation is addition. Of course, an Euler operator does not make clear exactly where in a given complex the change is effected, but only says how the configuration is changed. In the same manner can different isomorphism classes of complexes have the same configuration.

Let us review now the process of constructing a cw-complex. The initial step is to produce V_0 0-cells. We assume that they are obtained one after the other. This means that each individual 0-cell increases the number of existing 0-cells by one, and also the number of connected components ($= b_0$) by one. In other words, the creation of one 0-cell is effected by the Euler-Poincaré operator $L_0 = X_0 + Y_0$. The next step is to introduce 1-cells by joining 0-cells. There are two possible effects on the Betti numbers: either b_0 is reduced by one, or a loop is formed. In the latter case, b_1 is increased by one. The first case is effected by the Euler-Poincaré operator $A_1 := X_1 - Y_0$ (make loop, join parts: mljp) which can be also expressed as $K_1 - L_0$. The formation of a loop is effected by $B_1 := X_1 + Y_1 = K_1 + L_1$ (make edge, make loop: meml).

Remark 5.1. *Note that each element of the basis K_i, L_j for $\mathfrak{E}$ found in Section 4 involves the creation or anihilation of a 0-cell due to the presence of the term X_0. The discussion above, however, asks for a different basis for the Euler-Poincaré operators. This is the content of Proposition 5.2. Note, however, that any $\mathbb{Z}$-linear combination of the K_i leaves the Betti numbers of a given complex unchangend.*

Proposition 5.2. *A basis for $\mathfrak{E}$ is given by*

$$A_i := X_i - Y_{i-1}, \quad B_0 := X_0 + Y_0, \quad B_j := X_j + Y_j \quad (i, j = 1, \dots, n) \tag{8}$$

Proof. The basis (8) is obtained from the K_i, L_j by addition and subtraction:

$$A_i = K_i - L_{i-1}, \quad B_j = K_j + L_j \quad (i, j = 1, \dots, n),$$

which are valid steps in the Gauss algorithm for solving systems of integer linear equations. □

Example 5.3. *Figure 5 illustrates some examples of $L_0 = Y_0 + X_0$ (make vertex, make component: mvmc), $L_1 = Y_1 - X_0$ (kill vertex, make loop: kvml), $L_2 = Y_2 + X_0$ (make vertex, make shell: mvms) as well as $K_1 = X_1 + X_0$ (make vertex, make edge: mvme) and $K_2 = X_2 - X_0$ (kill vertex, make area: kvma). The example with L_2 takes the Betti numbers $b_0 = b_1 = 1$, $b_2 = 0$ to $b_0 = b_1 = b_2 = 1$. The example with K_2 takes $b_0 = b_1 = 1$ to $b_0 = b_1 = 1$.*

6. Euler-Poincaré operators as polynomials

In this section we give an alternative description of Euler-Poincaré operators. It is based on the observation that A_i and A_{i+1} differ only by an index shift. The same observation holds true for the B_j. Algebraically, this can be described using polynomials in one variable X in the following way. First set $X := X_1$, and then make the identifications:

$$1 = X^0 := X_0, \quad X^i := X_i, \quad X^{n+i+1} := Y_i \quad (i = 1, \dots, n) \tag{9}$$

In this way, the elements of V are identified with polynomials in X. As terms of degree higher than $2n + 1$ are ignored, we can say that the identifications yield a bijective map

$$f \colon V \to \mathbb{Z}[X]_{2n+1} =: \mathcal{V} \subseteq \mathbb{Z}[X] =: \mathcal{R}$$

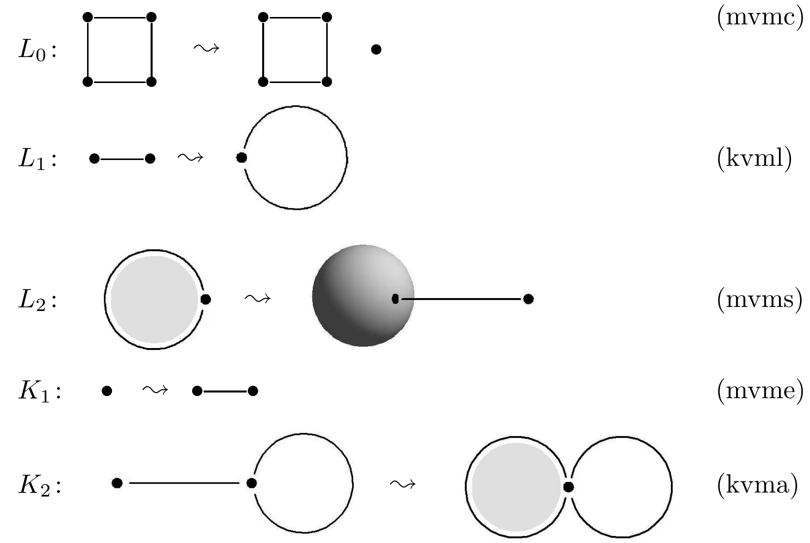


FIGURE 5. Examples of Euler-Poincaré operators.

Here, $\mathcal{V}$ is the submodule of $\mathbb{Z}[X]$ consisting of polynomials of degree $\leq 2n+1$. Via the identification map f, elements of $\mathcal{V}$ can be multiplied like integer polynomials, a fact which we will exploit in the sequel. We also call the elements of $f(\mathfrak{E})$ Euler-Poincaré operators and identify them with their preimages in $\mathfrak{E}$.

The basis A_i, B_j of $\mathfrak{E}$ as a $\mathbb{Z}$-module can now be expressed as:

$$A_i = X^i - X^{n+i} = X^i \cdot (1 - X^n), \qquad (i = 1, \ldots, n) \tag{10}$$

$$B_j = X^j + X^{n+j+1} = X^j \cdot (1 + X^{n+1}), \qquad (j = 0, \ldots, n) \tag{11}$$

Hence, general Euler-Poincaré operators can be built up from the two "fundamental" operators $A := 1 - X^n$ and $B := 1 + X^{n+1}$ and a "dimension shift" using polynomial multiplication. In other words, any arbitrary Euler-Poincaré operator is an $\mathcal{R}$-linear combination of AX and B.

Note that there is a very simple, yet effective way of testing whether a given polynomial $F \in \mathcal{V}$ lies in $f(\mathfrak{E})$ or not.

Proposition 6.1. *Let* $F = \sum_{i=0}^{2n+1} \alpha_i X^i \in \mathcal{V}$.

(1) *Assume n even. Then $F \in f(\mathfrak{E})$ if and only if $F(-1) = 0$.*
(2) *Assume n odd. Then $F \in f(\mathfrak{E})$ if and only if*

$$\sum_{i=0}^{n} (-1)^i \alpha_i = \sum_{i=0}^{n} (-1)^i \alpha_{n+1+i}.$$

Proof. (1) holds true by the observation for n even:

$$\begin{aligned} F(-1) &= \sum_{i=0}^{n}(-1)^i\alpha_i - \sum_{i=0}^{n}(-1)^i\alpha_{n+1+i} \\ \Leftrightarrow \sum_{i=0}^{n}(-1)^i\alpha_i &= \sum_{i=0}^{n}(-1)^i\alpha_{n+1+i} \end{aligned}$$

The latter equality is nothing but the Euler-Poincaré formula.

(2) is obvious. □

The problem of calculating the coefficients of $F \in f(\mathfrak{E})$ with respect to the basis A_i, B_j can be solved by the Gauss elimination algorithm for systems of linear equations. The structure, however of A_i and B_j in terms of polynomials allows for finding explicit formulae for the expansion of F. To this end, let $\mathfrak{I}$ be the following submodule of $\mathcal{V}$:

$$\mathfrak{I} := \bigoplus_{i=1}^{n}\mathbb{Z}A_i \oplus \bigoplus_{j=0}^{n}\mathbb{Z}B_j \tag{12}$$

It is the direct sum of the two submodules $\mathfrak{A} = \bigoplus_{i=1}^{n}\mathbb{Z}A_i$ and $\mathfrak{B} = \bigoplus_{j=0}^{n}\mathbb{Z}B_j$.

Lemma 6.2. *The following statements hold true:*

$$\mathfrak{I} = f(\mathfrak{E}) \tag{13}$$

$$A \in \mathfrak{I} \Leftrightarrow n \textit{ is even} \tag{14}$$

$$X^{n+1}A \in \mathfrak{I} \Leftrightarrow n \textit{ is even} \tag{15}$$

Proof. The first statement holds true by what has been said abve.

As to the second. we have

$$A = 1 - X^n = \begin{cases} -K_n, & n \text{ even} \\ K_n - 2X_n, & n \text{ odd} \end{cases}.$$

By (13), $K_n \in \mathfrak{I}$. Also, $K_n - 2X_n \notin \mathfrak{I}$, as otherwise $2X_n = K_n - A \in \mathfrak{I}$ by (13). Hence, we are done.

The third statement:

$$X^{n+1}A = X^{n+1} - X^{2n+1} = Y_0 - Y_n = \begin{cases} L_0 - L_n, & n \text{ even} \\ L_0 - L_n - 2X_0, & n \text{ odd} \end{cases}$$

By the same reasoning as before, the assertion follows. □

The structural result of Proposition 6.1 is that for n even,

$$F \in \mathfrak{I} \Leftrightarrow F \text{ is an } \mathcal{R}\text{-multiple of } X+1.$$

For n odd, the result is in fact quite similar:

Lemma 6.3. *Let n be odd, and $F = \sum_{i=0}^{2n+1}\alpha_i X^i \in \mathbb{Z}[X]$. Then*

$$F \in \mathfrak{I} \Leftrightarrow F - 2\sum_{i=0}^{n}(-1)^i\alpha_i \textit{ is divisible in } \mathbb{Z}[X] \textit{ by } X+1.$$

Proof. Observe that for n odd

$$F(-1) - 2\sum_{i=0}^{n}(-1)^i\alpha_i = \sum_{i=0}^{n}(-1)^i\alpha_{n+1+i} - \sum_{i=0}^{n}(-1)^i\alpha_i = 0$$
$$\Leftrightarrow \sum_{i=0}^{n}(-1)^i\alpha_i = \sum_{i=0}^{n}(-1)^i\alpha_{n+1+i}$$

from which the assertion follows. □

For later use, the following formulae are interesting:

Lemma 6.4. *It holds true that*

$$1 + X = X(1 - X^n) + (1 + X^{n+1}) \tag{16}$$
$$X^{n+j}(1 + X) = X^j(1 + X^{n+1}) - X^j(1 - X^n) \tag{17}$$

for $j = 1, \ldots, n$.

Proof. Direct computations yield the formulae. □

6.1. **Expansion for n even.** In this subsection, we assume that n is an even natural number.

Lemma 6.5. *The following identities hold true for n even:*

$$1 - X^n = (1 + X)(1 - X + X^2 - + \cdots - X^{n-1}) \tag{18}$$
$$1 + X^{n+1} = (1 + X)(1 - X + X^2 - + \cdots + X^n) \tag{19}$$

Proof. Direct computation yields the identities for n even. □

Example 6.6. *In Lemma 6.2, we have seen that $1 - X^n \in \mathfrak{I}$. The identities (18) and (16) yield the expansion with respect to the basis A_i, B_j:*

$$1 - X^n = (1 + X)\sum_{j=0}^{n-1}(-1)^j X^j$$
$$= \sum_{i=1}^{n}(-1)^{i-1}A_i + \sum_{j=0}^{n-1}(-1)^j B_j$$

From this, formula (17) yields the coefficients for $X^{n+1}(1 - X^n) \in \mathfrak{I}$ with respect to the basis A_i, B_j:

$$X^{n+1}(1 - X^n) = \sum_{i=1}^{n}(-1)^i A_i + \sum_{j=1}^{n}(-1)^j B_j \tag{20}$$

Proposition 6.7. *Let $F(X) = \sum_{i=0}^{2n+1} \alpha_i X^i \in \mathbb{Z}[X]$ with n even. Then $F(X)$ has the expansion*

$$F(X) = \beta_0 + \sum_{i=1}^{n}\gamma_i A_i + \beta_1 B_0 + \sum_{j=1}^{n}\delta_j B_j$$

with

$$\beta_0 = F(-1), \quad \beta_1 = -\sum_{\ell=1}^{2n+1} (-1)^\ell \alpha_\ell = \alpha_0 - \beta_0$$
$$\gamma_i = (-1)^i \sum_{\ell=i}^{n} (-1)^\ell (\alpha_\ell - \alpha_{n+1+\ell})$$
$$\delta_j = \alpha_j - \gamma_j$$

for $i, j = 1, \dots, n$.

Proof. First observe that

$$F(X) = (1+X) \sum_{i=1}^{2n+1} \beta_i X^{i-1} + \beta_0,$$

where

$$\beta_i = (-1)^i \sum_{\ell=i}^{2n+1} (-1)^\ell \alpha_\ell, \quad i = 0, \dots, 2n+1.$$

Notice that $\beta_0 = F(-1)$. Now, using the formulae in Lemma 6.4, we find

$$\begin{aligned}
F(X) - \beta_0 &= \sum_{i=1}^{n+1} \beta_i X^{i-1}(1 + X^{n+1}) + \sum_{i=1}^{n+1} \beta_i X^i (1 - X^n) \\
&+ \sum_{i=n+2}^{2n+1} \beta_i X^{i-1}(1+X) \\
&= \sum_{i=0}^{n} \beta_{i+1} X^i (1 + X^{n+1}) + \sum_{i=1}^{n} \beta_i X^i (1 - X^n) + \beta_{n+1} X^{n+1}(1 - X^n) \\
&+ \sum_{i=2}^{n+1} \beta_{n+i} X^{i-1}(1 + X^{n+1}) - \sum_{i=2}^{n+1} \beta_{n+i} X^{i-1}(1 - X^n) \\
&= \beta_1 (1 + X^{n+1}) + \beta_{n+1} X^{n+1} (1 - X^n) \\
&+ \sum_{i=1}^{n} (\beta_{i+1} + \beta_{n+i+1}) X^i (1 + X^{n+1}) + \sum_{i=1}^{n} (\beta_i - \beta_{n+i+1}) X^i (1 - X^n)
\end{aligned}$$

Together with equation (20), this yields

$$F(X) = \beta_0 + \sum_{i=0}^{n} (\beta_i - \beta_{i+n+1} + (-1)^i \beta_{n+1}) A_i \tag{21}$$

$$+ \beta_1 B_0 + \sum_{i=0}^{n} (\beta_{j+1} + \beta_{j+n+1} + (-1)^j \beta_{n+1}) B_j \tag{22}$$

Simplifying the coefficients somewhat yields for $i, j = 1, \dots, n$:

$$\begin{aligned}
\gamma_i &:= \beta_i - \beta_{i+n+1} + (-1)^i \beta_{n+1} \\
\delta_j &:= \beta_{j+1} + \beta_{n+j+1} + (-1)^{j+1} \beta_{n+1} \\
&= \alpha_j - \beta_j + \beta_{n+j+1} - (-1)^j \beta_{n+j} \\
&= \alpha_j - \gamma_j
\end{aligned}$$

Up to here, no specific use of the fact has been made that n is even. However, notice that for n even:

$$
\begin{aligned}
(-1)^i\beta_{n+1} - \beta_{n+1+i} &= (-1)^i(-1)^{n+1}\sum_{\ell=n+1}^{2n+1}(-1)^\ell\alpha_\ell \\
&\quad - (-1)^{n+1+i}\sum_{\ell=n+1+i}^{2n+1}(-1)^\ell\alpha_\ell \\
&= (-1)^{i+1}\sum_{\ell=n+1}^{n+i}(-1)^\ell\alpha_\ell
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\gamma_i &= \beta_i - (-1)^i\sum_{\ell=n+1}^{n+i}(-1)^\ell\alpha_\ell \\
&= (-1)^i\sum_{\ell=i}^{n}(-1)^\ell\alpha_\ell + (-1)^i\sum_{\ell=n+i+1}^{2n+1}(-1)^\ell\alpha_\ell \\
&= (-1)^i\sum_{\ell=i}^{n}(-1)^\ell\alpha_\ell + (-1)^i\sum_{\ell=i}^{n}(-1)^{\ell+n+1}\alpha_{n+1+\ell} \\
&= (-1)^i\sum_{\ell=i}^{n}(-1)^\ell(\alpha_\ell - \alpha_{n+1+\ell})
\end{aligned}
$$

which proves the assertion. □

Remark 6.8. *Notice that $\delta_n = \alpha_{2n+1}$. This follows from the fact that among the polynomials A_i, B_j only B_n contains a term of highest degree (i.e. an X^{2n+1}-term). Comparing the coefficients then yields the asserted statement. Alternatively, a direct computation yields the same result.*

Example 6.9. *The polynomial $1 - X^n$ with n even has coefficients*

$$
\alpha_i = \begin{cases} 1, & i = 0 \\ -1, & i = n \\ 0, & \text{otherwise.} \end{cases}
$$

Hence,

$$
\begin{aligned}
&\beta_0 = 0, \quad \beta_1 = 1 \\
&\gamma_i = (-1)^i(-1)^n\alpha_n = (-1)^{i+1} && (i = 1, \dots, n) \\
&\delta_j = -\gamma_j = (-1)^j && (j = 1, \dots, n-1) \\
&\delta_n = -1 - (-1)^{n+1} = 0 && (\text{or by Remark 6.8})
\end{aligned}
$$

This coincides with the expansion in Example 6.6.

6.2. **Expansion for n odd.** Here, we assume that n is an odd natural number, and again let $F = \sum_{i=0}^{2n+1}\alpha_i X^i \in \mathbb{Z}[X] = \mathfrak{R}$ be a fixed polynomial. By Lemma 6.3, we cannot ignore the constant term if n is odd. This is in contrast to the case of n

even. A general element of $\mathfrak{I}$ is an $\mathcal{R}$-linear combination of $1-X^n$ and $1+X^{n+1}$. Therefore, if G, H are polynomials in $\mathcal{R}$, let

$$\mathcal{R}(G,H)$$

denote the space of all $\mathcal{R}$-linear combinations of G and H. It is clearly an $\mathcal{R}$-module inside $\mathcal{R}$ and is called the *ideal* in $\mathcal{R}$ *generated by* G *and* H. Hence,

$$\mathfrak{I} \subseteq I := \mathcal{R}(1-X^n, 1+X^{n+1}), \tag{23}$$

because any $\mathbb{Z}$-linear combination of A_i, B_j can be expressed as

$$G \cdot (1-X^n) + H \cdot (1+X^{n+1})$$

for polynomials $G, H \in \mathcal{R}$.

If, on the other hand, F is given as an $\mathcal{R}$-linear combination of $1-X^n$ and $1+X^{n+1}$, there is a simple way of checking whether $F \in \mathfrak{I}$ or not:

Lemma 6.10. *It holds true for n odd and $a, b \in \mathbb{Z}$ that*

$$a(1-X^n) + bX^{n+1}(1-X^n) \in \mathfrak{I} \Leftrightarrow a = b.$$

Proof. Observe that

$$A_0B_0 = 1 - X^n + X^{n+1}(1-X^n) = 1 + X^{n+1} - X^n(1+X^{n+1}) \in \mathfrak{I}. \tag{24}$$

Hence, the implication $\Leftarrow$ is obvious.

Assume now that $G_{a,b} := a(1-X^n) + bX^{n+1}(1-X^n) \in \mathfrak{I}$. Then also

$$G_{a,b} - b \cdot G_{1,1} = (a-b)(1-X^n) \in \mathfrak{I}.$$

By Lemma 6.2 (14), this implies $a = b$, and the implication $\Rightarrow$ is proven. □

Lemma 6.11. *Let n odd. Then*

$$I = \mathcal{R}(1-X^n, 1+X^{n+1}) = \mathcal{R}(2, 1+X).$$

Proof. The first equality is clear. Polynomial division yields:

$$1 + X^{n+1} = -X(1-X^n) + 1 + X \tag{25}$$

$$1 - X^n = (-1 + X - X^2 + - \cdots - X^{n-1})(1+X) + 2 \tag{26}$$

By (25),

$$\mathcal{R}(1-X^n, 1+X^{n+1}) = \mathcal{R}(1-X^n, 1+X),$$

whereas by (26),

$$\mathcal{R}(1-X^n, 1+X) = \mathcal{R}(2, 1+X)$$

which concludes the proof. □

As in the case n even, we define for $i = 0, \dots, 2n+1$:

$$\beta_i = (-1)^i \sum_{\ell=i}^{2n+1} (-1)^\ell \alpha_\ell$$

Remark 6.12. *By Lemma 6.11, F lies in the ideal I if and only if F is an $\mathcal{R}$-linear combination of 2 and $1+X$. The latter is equivalent to*

$$F = G \cdot (1+X) + 2a,, \tag{27}$$

where $F(-1) = 2a$ is an even integer. In that case, it is clear that $F \in \mathfrak{I}$ if and only if $a = \sum\limits_{i=0}^{n} (-1)^i \alpha_i$ or, equivalently, $a = \beta_{n+1}$.

Proposition 6.13. *Let* $F(X) = \sum_{i=0}^{2n+1} \alpha_i X^i \in \mathfrak{I}$ *with* n *odd. Then* $F(X)$ *has the expansion*

$$F(X) = \sum_{i=1}^{n} \gamma_i A_i + \alpha_0 B_0 + \sum_{j=1}^{n-1} \delta_j B_j + \alpha_{2n+1} B_n$$

with

$$\gamma_i = (-1)^i \sum_{\ell=i}^{n} (-1)^\ell (\alpha_\ell - \alpha_{\ell+n+1}) \qquad (i = 1, \ldots, n)$$
$$\delta_j = \alpha_j - \gamma_j \qquad (j = 1, \ldots, n-1)$$

Proof. As in the proof of Proposition 6.7, we obtain

$$F(X) = 2\beta_{n+1} + \beta_{n+1} X^{n+1} A_0 + \sum_{i=1}^{n} (\beta_i - \beta_{n+1+i}) A_i + \beta_1 B_0 + \sum_{i=1}^{n} (\beta_{i+1} + \beta_{n+1+i}) B_i$$

(25) and (26) yield

$$2 = A_0 + \sum_{i=1}^{n} (-1)^{i+1} A_i + \sum_{i=0}^{n-1} (-1)^i B_i \tag{28}$$

This means that $2\beta_{n+1} + \beta_{n+1} X^{n+1} A_0$ contains the term $\beta_{n+1}(A_0 + X^{n+1} A_0) = \beta_{n+1} A_0 B_0$. Hence, by equation (24) it holds true that

$$\begin{aligned} F(X) &= \beta_{n+1} A_0 B_0 + \sum_{i=1}^{n} (\beta_i - \beta_{n+1+i} + (-1)^{i+1} \beta_{n+1}) A_i \\ &\quad + (\beta_1 + \beta_{n+1}) B_0 + (\beta_{n+1} + \beta_{2n+1}) B_n \\ &\quad + \sum_{i=1}^{n-1} (\beta_{i+1} + \beta_{n+1+i} + (-1)^i \beta_{n+1}) B_i \\ &= \sum_{i=1}^{n} (\beta_i - \beta_{n+1+i} + (-1)^{i+1} \beta_{n+1}) A_i \\ &\quad + (\beta_1 + 2\beta_{n+1}) B_0 + \beta_{2n+1} B_n \\ &\quad + \sum_{i=1}^{n-1} (\beta_{i+1} + \beta_{n+1+i} + (-1)^i \beta_{n+1}) B_i \end{aligned}$$

Notice that $\beta_{2n+1} = \alpha_{2n+1}$. We now set

$$\gamma_i := \beta_i - \beta_{n+1+i} + (-1)^{i+1} \beta_{n+1} \qquad (i = 1, \ldots, n)$$
$$\delta_j := \beta_{j+1} + \beta_{n+1+j} + (-1)^j \beta_{n+1} \qquad (j = 1, \ldots, n-1)$$

and obtain

$$\delta_j = \alpha_j - \gamma_j$$

and also

$$\begin{aligned}\beta_i - \beta_{n+1+i} &= (-1)^i \sum_{\ell=i}^{2n+1} (-1)^\ell \alpha_\ell - (-1)^{n+1+i} \sum_{\ell=n+1+i}^{2n+1} (-1)^\ell \alpha_\ell \\ &= (-1)^i \sum_{\ell=i}^{n+i} (-1)^\ell \alpha_\ell\end{aligned}$$

which yields for $i = 1, \dots, n$

$$\begin{aligned}\gamma_i &= (-1)^i \sum_{\ell=i}^{n+i} (-1)^\ell \alpha_\ell - (-1)^i \sum_{\ell=n+1}^{2n+1} (-1)^\ell \alpha_\ell \\ &= (-1)^i \sum_{\ell=i}^{n} (-1)^\ell \alpha_\ell - (-1)^i \sum_{\ell=n+1+i}^{2n+1} (-1)^\ell \alpha_\ell \\ &= (-1)^i \sum_{\ell=i}^{n} (-1)^\ell \alpha_\ell - (-1)^i \sum_{\ell=i}^{n} (-1)^{\ell+n+1} \alpha_{\ell+n+1} \\ &= (-1)^i \sum_{\ell=i}^{n} (-1)^\ell (\alpha_\ell - \alpha_{\ell+n+1})\end{aligned}$$

which proves the assertion. □

7. Euler-Poincaré operators in dimension $n \leq 4$

In this section, we write down explicitly the low-dimensional Euler-Poincaré operators in their expansions as calculated in Section 6.

7.1. **The case $n = 1$.** Euler-Poincaré operators in dimension $n = 1$ are modifications of graphs. By Lemma 6.3, they can be represented by integer polynomials F of degree $\leq 2n + 1 = 3$ such that $F(-1)$ is an even integer. Also, $A = 1 - X$ is not an Euler-Poincaré operator by Lemma 6.2. Observe that here,

$$A = 1 - X = X_0 - X_1$$

adds a vertex and removes an edge, and that this operation cannot leave the Betti numbers unchanged (which it must, if A were an Euler-Poincaré operator). However,

$$\mathfrak{I} = \mathbb{Z}X(1 - X) \oplus \mathbb{Z}(1 + X^2) \oplus \mathbb{Z}X(1 + X^2)$$

from which we obtain the elementary operators

$X - X^2 = X_1 - Y_0$	make edge, kill component
$1 + X^2 = X_0 + Y_0$	make vertex and component
$X + X^3 = X_1 + Y_1$	make edge and loop

Note that in the literature, "kill component" or "make component" does not seem to be considered very often. Figure 6 illustrates the first and the third elementary Euler-Poincaré operator.

$X - X^2$:

$X + X^3$:

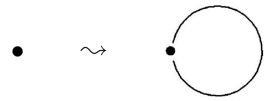


FIGURE 6. Elementary Euler-Poincaré operators in dimension 1.

A general polynomial $F = \alpha_0 + \alpha_1 X + \alpha_2 X^2 + \alpha_3 X^3$ has the expansion

$$F = (\alpha_1 - \alpha_3)A_1 + \alpha_0 B_0 + \alpha_3 B_1,$$

as can be readily checked, also without resorting to Proposition 6.13.

7.2. **The case** $n = 2$**.** In dimension 2, the polynomials $F = \alpha_0 + \alpha_1 X + \alpha_2 X^2 + \alpha_3 X^3 + \alpha_4 X^4 + \alpha_5 X^5 \in \mathbb{Z}[X]$ have the expansion

$$\begin{aligned} F = F(-1) &+ (\alpha_1 - \alpha_2 - \alpha_4 + \alpha_5)A_1 + (\alpha_2 - \alpha_5)A_2 \\ &+ (\alpha_0 - F(-1))B_0 + (\alpha_2 + \alpha_4 - \alpha_5)B_1 + \alpha_5 B_2 \end{aligned}$$

with $F \in \mathfrak{I}$ if and only if $F(-1) = 0$. We have

$$\mathfrak{I} = \mathbb{Z}(1 - X^2) \oplus \mathbb{Z}(X - X^3) \oplus \mathbb{Z}(1 + X^3) \oplus \mathbb{Z}(X + X^4)$$

with the elementary operators

$1 - X^2 = X_0 - X_2$	make vertex, kill area
$X - X^3 = X_1 - Y_0$	make edge, kill component
$1 + X^3 = X_0 + Y_0$	make vertex and component
$X + X^4 = X_1 + Y_1$	make edge and loop
$X^2 + X^5 = X_2 + Y_2$	make area and shell

Figure 7 depicts the first elementary Euler-Poincaré operator. The last elementary operator can be imagined as attaching to a given vertex a "bubble", resulting in an extra surface which together with the vertex circumscribes some space: a shell.

$1 - X^2$:

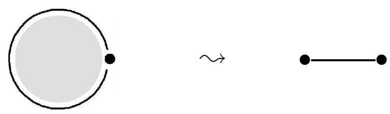


FIGURE 7. An elementary Euler-Poincaré operator in dimension 2.

On the other hand, from the operator $1+X = X_0+X_1$ we obtain the fundamental Euler-Poincaré operators $1 - X^2$ and $1 + X^3$ by polynomial multiplications:

$$1 - X^2 = (1 + X)(1 - X), \quad 1 + X^3 = (1 + X)(1 - X + X^2)$$

from which the elementary operators are easily derived by multiplication with powers of X.

7.3. **The case $n = 3$.** In the three-dimensional case, the expansion for $F \in \mathfrak{I}$ is

$$F = \gamma_1 A_1 + \gamma_2 A_2 + \gamma_3 A_3 + \alpha_0 B_0 + \delta_1 B_1 + \delta_2 B_2 + \alpha_7 B_3$$

with

$$\begin{aligned}
\gamma_1 &= (\alpha_1 - \alpha_5) - (\alpha_2 - \alpha_6) + (\alpha_3 - \alpha_7)\\
\gamma_2 &= (\alpha_2 - \alpha_6) - (\alpha_3 - \alpha_7)\\
\gamma_3 &= \alpha_3 - \alpha_7\\
\delta_1 &= \alpha_5 + (\alpha_2 - \alpha_6) - (\alpha_3 - \alpha_7)\\
\delta_2 &= \alpha_6 + \alpha_3 - \alpha_7
\end{aligned}$$

The elementary operators are

$X - X^4 = X_1 - Y_0$	make edge, kill component
$X^2 - X^5 = X_2 - Y_1$	make area, kill loop
$X^3 - X^6 = X_3 - Y_2$	make volume, kill shell
$1 + X^4 = X_0 + Y_0$	make vertex and component
$X + X^5 = X_1 + Y_1$	make edge and loop
$X^2 + X^6 = X_2 + Y_2$	make area and shell
$X^3 + X^7 = X_3 + Y_3$	make volume and 3-hole

The latter Euler-Poincaré operator does not seem to be encountered in the literature very often, and can be imagined as the three-dimensional analogon of the operator "make area and shell" described in Section 7.2.

7.4. **The case $n = 4$.** We refrain from giving explicit lists of expansions in the case of dimension 4. However, we note that the fundamental Euler-Poincaré operators are again multiples of $1 + X$ in the same way as for $n = 2$:

$$\begin{aligned}
1 - X^4 &= (1 + X)(1 - X + X^2 - X^3)\\
1 + X^5 &= (1 + X)(1 - X + X^2 - X^3 + X^4).
\end{aligned}$$

8. Topological architectural complexes

Let us review the inductive construction of a cw-complex:

(1) Take a set X_0 whose points are considered as 0-cells.
(2) Glue 1-cells by mapping their boundary points into X_0. This yields X_1.
(3) Glue 2-cells e by mapping their boundary into X_1 using characteristic maps $\chi_e \colon S^1 \to X^1$.
(4) Etc.

Assume $m > 1$. The degree of the characteristic maps projected to the spheres

$$S^m \xrightarrow{\chi_e} X^m \longrightarrow X^m/(X^m \setminus e') \cong S^m$$

yields for any $m+1$-cell e the entry $d = \delta_{ee'}$ of the matrix representing the boundary operator of the associated chain complex.

The whole concept works for cw-complexes because of the fact that the homology group $H_m(S^m, \mathbb{Z}) \cong \mathbb{Z}$, i.e. the m-sphere has free m-th homology of rank one, just like any closed connected orientable manifold has. The composed map above sends a fixed generator γ of $H_m(S^m, \mathbb{Z})$ to a multiple: $\gamma \mapsto d \cdot \gamma$, and this yields the matrix entry $\delta(e, e')$.

In the following subsections, we will apply the glueing process to more general manifolds with boundary which allow to obtain coefficients for the partial boundary operator of a certain type of architectural complex.

8.1. **Topological realisation of architectural complexes.**

Definition 8.1. *Let n be a natural number. A* topological $n+1$-region *is a connected orientable topological manifold of dimension $n+1$ whose boundary is homeomorphic to the disjoint union of a finite number of n-dimensional closed connected orientable manifolds.*

Remark 8.2. *A topological 1-region is either a loop or a 1-cell. An $n+1$-sphere is a topological $n+1$-region.*

Definition 8.3. *A* topological architectural complex *is the space $\mathcal{X} = (X, \chi)$ obtained by the following inductive glueing process:*

(1) *Take a finite set X_0 of 0-cells.*
(2) *Glue to X_0 some finitely many topological 1-regions b along their boundaries:*
$$\chi_b \colon \partial b \to X_0$$
and obtain the space X_1.
(3) *Glue to X_1 some finitely many topological 2-regions c along their boundaries:*
$$\chi_c \colon \partial c \to X_1$$
and obtain X_2.
(4) *Etc. End after n steps with $X = X_n$.*

Denote by B the set of all regions appearing in the process, and $\chi = (\chi_b)_{b \in B}$.

As usual, we define B_m to be the set of topological m-regions (or m-cells) of $\mathcal{X}$, and X_m will be called the *m-skeleton* of $\mathcal{X}$. Let $b \in B_{n+1}$. Given some $b' \in B_n$, we have a map from its boundary
$$\chi_{bb'} \colon \partial b \to X_n \to X_n/(X_n \setminus b') =: X_{b'},$$
where $X_n/(X_n \setminus b')$ is the quotient space obtained by identifying all points of X_n in the complement of b'. This map descends to a linear map on homology
$$\varphi_{bb'} \colon H_n(\partial b, \mathbb{Z}) \to H_n(X_{b'}, \mathbb{Z}),$$
and by [4, Theorem 3.26] it holds true that

$$H_n(X_{b'}, \mathbb{Z}) \cong \mathbb{Z} \tag{29}$$
$$H_n(\partial b, \mathbb{Z}) \cong \mathbb{Z}^r, \tag{30}$$

where r is the number of connected components of ∂b. A set of generators for $H_n(\partial b, \mathbb{Z})$ is given by the fundamental classes $x_1, \ldots, x_r$ for the individual components of ∂b. Denote the fundamental class of X_b by x. Then it holds true that

$$\varphi_{bb'}(x_i) = \alpha_i x \tag{31}$$

for some $\alpha_i \in \mathbb{Z}$.

Definition 8.4. *The* degree $\delta(b,b')$ *of* $\chi_{bb'}$ *is defined as*

$$\delta(b,b') := \sum_{i=1}^{r} \alpha_i,$$

where the $\alpha_i \in \mathbb{Z}$ *are given by equation (31).*

Note that if $\mathbb{X}$ is a cw-complex, then the degree $\delta(b,b')$ coincides with the (b,b')-entry of the boundary operator for its associated chain complex. The definition above allows now to define to any given topological architectural complex $\mathcal{X}$ a chain complex and an architectural complex.

Proposition 8.5. *Let* $\mathcal{X}$ *be a topological architectural complex. Then* $\delta :\subseteq B \times B \to \mathbb{Z}$ *is the partial boundary matrix of an architectural complex* $\mathscr{C}$*, and the linear map*

$$\partial_{n+1} \colon C_{n+1} := \mathbb{Z}B_{n+1} \to \mathbb{Z}B_n =: C_n,\ b \mapsto \sum_{b' \in B_n} \delta(b,b') \cdot b'$$

yields a chain complex $\mathcal{C}$*. If* $\mathcal{X}$ *is cw, then* $\mathscr{C}$ *coincides with the architectural complex associated to that cw-complex. The like holds for the chain complex* $\mathcal{C}$*.*

Proof. We need only to show

$$\partial_n \circ \partial_{n+1} = 0 \tag{32}$$

All asserted statements follow from this. We will derive (32) from Lemma 8.6 and Lemma 8.7 below. □

In order to ease the notation, we will omit in what follows the coefficient ring $\mathbb{Z}$ in the notation for (relative and absolute) homology groups.

Lemma 8.6. *Let* $\mathcal{X}$ *be a topological architectural complex. It holds true that*

$$H_n(X_n, X_{n-1}) \cong C_n$$

Proof. This follows from the observation that X_n/X_{n-1} is a wedge sum of closed orientable n-dimensional manifolds, one for each n-region of $\mathcal{X}$. □

Lemma 8.7. *The following diagram is commutative:*

$$\begin{array}{ccc} H_{n+1}(X_{n+1}, X_n) & \xrightarrow{\quad \partial_{n+1} \quad} & H_n(X_n, X_{n-1}) \\ & {\scriptstyle d_{n+1}} \searrow \quad \nearrow {\scriptstyle j_n} & \\ & H_n(X_n) & \end{array}$$

where ∂_{n+1} *is defined as in Proposition 8.5 (using Lemma 8.6), the map* d_{n+1} *occurrs in the long exact homology sequence for the pair* (X_{n+1}, X_n)*, and* j_n *in the long exact sequence for* (X_n, X_{n-1})*.*

Proof. Let $b \in B_{n+1}$ arbitrary, and let $\bar{b}$ be the closure of the $n+1$-region b. The long exact sequence for the pair $(\bar{b}, \partial b)$ is given as

$$\cdots \longrightarrow H_{n+1}(\partial b) \longrightarrow H_{n+1}(\bar{b}) \longrightarrow H_{n+1}(\bar{b}, \partial b) \xrightarrow{d_{n+1}^b} H_n(\partial b) \longrightarrow \cdots$$

The other pairs (X_{n+1}, X_n) and (X_n, X_{n-1}) yield similarly long exact sequences containing the maps

$$\begin{array}{ccccc} & & H_{n+1}(X_{n+1}, X_n) & \xrightarrow{d_{n+1}} & H_n(X_n) \\ 0 & \longrightarrow & H_n(X_n) & \xrightarrow{j_n} & H_n(X_n, X_{n-1}) \end{array}$$

Pick some $b' \in B_n$. Together with the glueing maps for b and the induced map of pairs $\varphi_b\colon (\bar{b}, \partial b) \to (X_{n+1}, X_n)$, we obtain the following commutative diagram:

$$\begin{array}{ccccc} H_{n+1}(\bar{b}, \partial b) & \xrightarrow{d^b_{n+1}} & H_n(\partial b) & \xrightarrow{\varphi_{bb'}} & H_n(X_{b'}) \\ \downarrow{\scriptstyle \varphi_b} & & \downarrow{\scriptstyle \chi_b} & \nearrow{\scriptstyle \pi_{b'}} & \\ H_{n+1}(X_{n+1}, X_n) & \xrightarrow{d_{n+1}} & H_n(X_n) & & \\ & \searrow{\scriptstyle \Phi} & \downarrow{\scriptstyle j_n} & & \\ & & H_n(X_n, X_{n-1}) & & \end{array}$$

Note that $\varphi_{bb'}$ is a linear map

$$\varphi_{bb'} : \mathbb{Z}^r \to \mathbb{Z},\ e_i \mapsto \alpha_i$$

where r is the number of connected components of ∂b, and e_i is the i-th standard basis vector of $\mathbb{Z}^r$. Further, $H_{n+1}(\bar{b}, \partial b) \cong \mathbb{Z}b$, and $\pi_{b'}\colon C_n \to \mathbb{Z}$ is the projection onto the b'-th factor of $C_n = \mathbb{Z}B_n$.

We now want to prove that $\Phi = \partial_{n+1}$ holds true, where ∂_{n+1} is as defined in Proposition 8.5.

Observe that $d^b_{n+1}\colon \mathbb{Z}b \to \mathbb{Z}^r$ is the diagonal embedding $b \mapsto (1, \ldots, 1)$, possibly after changing some orientations of n-regions. Hence,

$$d^b_{n+1}(\varphi_{bb'}(b)) = \sum_{i=1}^{r} \alpha_i = \delta(b, b')$$

Next, $\varphi_b\colon \mathbb{Z}b \to \mathbb{Z}B_{n+1}$ is the map which makes $\mathbb{Z}b$ a direct summand of C_{n+1}. This means that

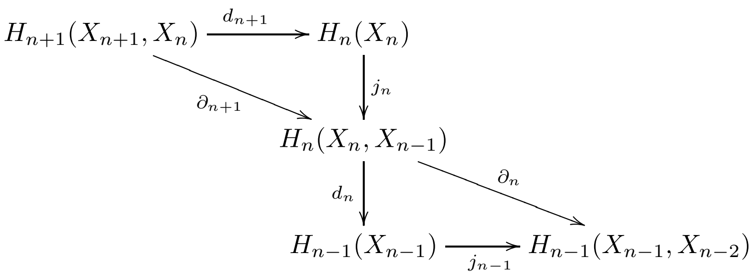
$$\pi_{b'}(\Phi(b)) = \pi_{b'}(\Phi(\varphi_b(b))) = d^b_{n+1}(\varphi_{bb'}(b)) = \delta(b, b')$$

which implies that

$$\Phi(b) = \sum_{b' \in B_n} \delta(b, b') \cdot b' = \partial_{n+1}(b),$$

as asserted. □

Proof of Prop. 8.5 (cont.) The long exact sequences for the three pairs (X_{n+1}, X_n), (X_n, X_{n-1}) and (X_{n-1}, X_{n-2}) yield the horizontal and vertical maps in the following commutative diagram with exact rows and columns.



The down-right arrows are given according to Lemma 8.7. As the map $\partial_{n+1} \circ \partial_n$ includes the two successive vertical maps from the long exact sequence for (X_n, X_{n-1}), it is zero. □

Remark 8.8. *The proof of Proposition 8.5 is in fact merely a careful analysis of a proof for the special case of cw-complexes as it can be found e.g. in* [4, §2.2].

8.2. **Topological homology.** In general, the Betti numbers of an architectural complex which is not a cw-complex do not correctly give the topological picture of the space represented by the architectural complex. For example, it is possible to construct a complex $\mathcal{X}$ having $b_0 = 0$. Topologically, this is senseless, as b_0 is usually interpreted as the number of connected components of $\mathcal{X}$.

Let $\mathcal{X} = (X, \chi)$ be an n-dimensional topological architectural complex and $\mathscr{C} = \mathscr{C}(X)$ its associated architectural complex. The aim of this subsection is to compare the two homologies $H_i(\mathcal{X}, \mathbb{Z}) := H_i(X)$ and $H_i(\mathscr{C}, \mathbb{Z})$.

Definition 8.9. $H_i^{\mathrm{top}}(\mathscr{C}, \mathbb{Z}) := H_i(\mathcal{X}, \mathbb{Z})$ *is called the* i-th topological homology group *of* $\mathscr{C}$*, and the rank of its free part the* i-th topological Betti number *of* $\mathscr{C}$*. The latter will be denoted as* $b_i^{\mathrm{top}}(\mathscr{C})$*.*

Note that the inclusion $X_k \subseteq X_{k+1}$ induces maps $\alpha_i \colon H_i(X_k) \to H_i(X_{k+1})$ between the homology groups.

Lemma 8.10. *For* $i \neq k$ *the maps* $\alpha_i \colon H_i(X_k) \to H_i(X_{k+1})$ *are all injective.*

Proof. We will show that the connection map d_{i+1} in the long exact sequence

$$\cdots \longrightarrow H_{i+1}(X_{k+1}, X_k) \xrightarrow{d_{i+1}} H_i(X_k) \xrightarrow{\alpha_i} H_i(X_{k+1}) \longrightarrow \cdots$$

for the pair (X_{k+1}, X_k) is the zero map. By exactness, it follows that α_i is injective.

If $i > k$, then $H_{i+1}(H_{k+1}, X_k) = 0$, whence the desired injectivity of α_i follows.

Assume now $i < k$. The relative homology group is defined through an exact sequence of chain complexes:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C_{i+1}(X_k) & \xrightarrow{d^{k}_{i+1}} & C_i(X_k) & \longrightarrow & C_{i-1}(X_k) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C_{i+1}(X_{k+1}) & \xrightarrow{d^{k+1}_{i+1}} & C_i(X_{k+1}) & \longrightarrow & C_{i-1}(X_{k+1}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C_{i+1}(X_{k+1}, X_k) & \xrightarrow{d^{k+1,k}_{i+1}} & C_i(X_{k+1}, X_k) & \longrightarrow & C_{i-1}(X_{k+1}, X_k) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By definition, d_{i+1} takes an element $[c] \in H_{i+1}(X_{k+1}, X_k)$ to $[\partial c] \in H_i(X_k)$ given as follows: it holds true that $d^{k+1,k}_{i+1}(c) = 0 \in C_i(X_{k+1}, X_k)$, i.e. is represented by an element $\partial c \in C_i(X_k)$. The homology class of ∂c in $H_i(X_k)$ is $d_{i+1}([c])$. However, the relative cycle c is represented by a chain $\gamma \in C_{i+1}(X_{k+1})$ such that $\partial c = d^{k+1}_{i+1}(\gamma) \in C_k(X_k)$. This means that all singular $i+1$-simplices $\Delta \to X_{k+1}$ in the support of γ are maps taking the sides of Δ already to X_k. Since $i < k$, the chain γ is homotopic to a chain γ' whose support consists of maps taking Δ already to X_k, i.e. $\gamma' \in C_{i+1}(X_k)$. Clearly, $d^k_{i+1}(\gamma') = d_{i+1}(\gamma)$. This implies $\partial c \in \operatorname{im} d^k_{i+1}$, i.e. $[\partial c] = 0 \in H_i(X_k)$. ☐

Theorem 8.11. *There is an inclusion map* $H_i(\mathscr{C}, \mathbb{Z}) \to H_i^{\text{top}}(\mathscr{C}, \mathbb{Z})$ *for all* $i \in \mathbb{N}$.

Proof. Consider the following diagram with exact rows and columns:

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
H_{i+1}(X_{i+1}, X_i) & \xrightarrow{d_{i+1}} & H_i(X_i) & \xrightarrow{\alpha_i} & H_i(X_{i+1}) \\
 & \searrow^{\partial_{i+1}} & \downarrow{\scriptstyle j_i} & & \\
 & & H_i(X_i, X_{i-1}) & & \\
 & & \downarrow{\scriptstyle d_i} & \searrow^{\partial_i} & \\
0 & \longrightarrow & H_{i-1}(X_{i-1}) & \xrightarrow[j_{i-1}]{} & H_{i-1}(X_{i-1}, X_{i-2})
\end{array}$$

By definition, it holds true that $H_i(\mathscr{C}, \mathbb{Z}) = \ker \partial_i / \operatorname{im} \partial_{i+1}$. From the injectivity of j_{i-1}, it follows by further inspecting the diagram that

$$\ker \partial_i = \ker d_i = \operatorname{im} j_i \cong H_i(X_i)$$

from which we obtain a map

$$\beta_i \colon \ker \partial_i \xrightarrow{\sim} H_i(X_i) \xrightarrow{\alpha_i} H_i(X_{i+1})$$

Observe that, by injectivity of j_i, the following holds true:

$$\ker \beta_i = j_i(\ker \alpha_i) = j_i(\operatorname{im} d_{i+1}) = \operatorname{im} \partial_{i+1}$$

Hence, the map β_i induces an inclusion map $H_i(\mathscr{C}, \mathbb{Z}) \to H_i(X_{i+1})$ and, by Lemma 8.10, the latter is a submodule of $H_i(X_n) = H_i^{\text{top}}(\mathscr{C}, \mathbb{Z})$. This proves the assertion for $i < n$. But in case $i = n$, the argumentation is simpler, because $H_{i+1}(X_{i+1}, X_i) = 0$: then we have

$$H_n(\mathscr{C}, \mathbb{Z}) = \ker \partial_n = \operatorname{im} j_n \cong H_n(X_n) = H_n^{\text{top}}(\mathscr{C}, \mathbb{Z})$$

also in this case. □

Corollary 8.12. *For all $i \in \mathbb{N}$ it holds true that $b_i(\mathscr{C}) \leq b_i^{\text{top}}(\mathscr{C})$.*

Proof. The inclusion $H_i(\mathscr{C}, \mathbb{Z}) \subseteq H_i^{\text{top}}(\mathscr{C}, \mathbb{Z})$ from Theorem 8.11 induces an inclusion between the free parts of the homology groups, which implies the assertion. □

8.3. **Computing topological Betti numbers.** The effect of an Euler-Poincaré-operator on a cw-complex is not always a cw-complex:

Example 8.13. *Let $n = 1$. Then the Euler-Poincaré operator $-1 - X^2 = -X_0 - Y_0$ removes the vertex from the cw-complex of a circle, resulting in a complex consisting of a single (circular) line. This architectural complex is not a cw-complex. At the same time, the corresponding Betti numbers change. However, the topological Betti numbers are equal in both cases. Figure 8 illustrates this effect and shows the corresponding Betti numbers.*

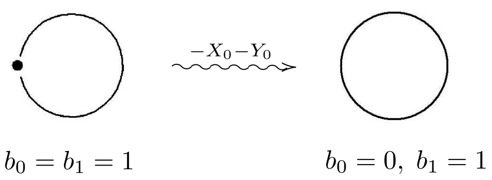


FIGURE 8. From cw- to non-cw-complex via Euler-Poincaré operator.

The positive lesson from this example is that applying Euler-Poincaré operators can possibly transform topological architectural complexes into cw-complexes without changing the topological Betti numbers. By doing this in a controlled manner, one can then compute the topological Betti numbers of a given architectural complex. Our starting point is the following fact:

Lemma 8.14. *Let $\mathscr{X}$ be a topological architectural complex which is cw. Then*

$$H_i(\mathscr{C}(\mathcal{X}), \mathbb{Z}) \cong H_i^{\text{top}}(\mathscr{C}(\mathcal{X}), \mathbb{Z})$$

for all $i \in \mathbb{N}$.

Proof. This is a well known fact from algebraic topology, cf. e.g. [4, Thm. 2.35]. □

Since we are concerned only with architectural complexes which are topological, we define:

Definition 8.15. *An architectural complex $\mathscr{C}$ is said to be* realisable*, if it is of the form $\mathscr{C} = \mathscr{C}(\mathcal{X})$, where $\mathcal{X}$ is a topological architectural complex.*

The following algorithm computes the topological Betti numbers of any given realisable architectural complex:

Algorithm 8.16. *The algorithm to follow transforms a given realisable architectural complex* $\mathscr{C} = \mathscr{C}(\mathcal{X})$ *into a cw-complex* $\mathrm{cw}(\mathscr{C})$ *with the property*

$$H_i(\mathrm{cw}(\mathscr{C}), \mathbb{Z}) \cong H_i^{\mathrm{top}}(\mathscr{C}, \mathbb{Z})$$

for all $i \in \mathbb{N}$. *Moreover, the algorithm yields a sequence of topological architectural complexes* $\mathcal{X}_k$ *for which*

$$H_i(\mathscr{C}, \mathbb{Z}) \subseteq H_i(\mathscr{C}(\mathcal{X}_k), \mathbb{Z}) \subseteq H_i(\mathscr{C}(\mathcal{X}_{k+1}), \mathbb{Z}) \subseteq H_i^{\mathrm{top}}(\mathscr{C}, \mathbb{Z})$$

and all $i \in \mathbb{N}$.

Input. A topological architectural complex $\mathcal{X}$ of dimension n.

Step 1. Make successively a minimal cellulisation of the 1-regions which are not already cells.

...

Step n. Make successively a minimal cellulisation of the n-regions which are not already cells.

Output. A cw-complex $\mathrm{cw}(\mathscr{C})$ for which

$$H_i(\mathrm{cw}(\mathcal{C}), \mathbb{Z}) \cong H_i^{\mathrm{top}}(\mathscr{C}, \mathbb{Z})$$

for all $i \in \mathbb{N}$.

Proof of correctness. Assume $\mathcal{X}_k$ already constructed and that

$$H_i(\mathscr{C}, \mathbb{Z}) \subseteq H_i(\mathscr{C}(\mathcal{X}_k), \mathbb{Z}) \subseteq H_i^{\mathrm{top}}(\mathscr{C}, \mathbb{Z})$$

holds true. Let B_m^k, C_m^k and X_m^k denote the m-regions, m-chains and m-skeleton of $\mathcal{X}_k$, respectively. W.l.o.g. we may assume there is some ℓ such that all $b \in B_m^k$ for all $m \leq \ell$ are in fact cells. Assume that $b \in B_{\ell+1}^k$ is a region which is not a cell. Then a cellulation of b imposes a cw-complex structure on the closure of b by introducing additional m-cells with $m \leq \ell$, and possibly replacing b by a disjoint union $b_1 \cup \cdots \cup b_r$ of $\ell + 1$-cells. This process yields a new topological architectural complex $\mathcal{X}_{k+1}$ together with inclusion maps

$$B_m^k \subseteq B_m^{k+1} \quad (m \leq \ell)$$
$$B_{\ell+1}^k \hookrightarrow C_{\ell+1}^{k+1}, \; b \mapsto b_1 + \cdots + b_r$$

Here, the latter map is meant to be the identity on $B_{\ell+i} \setminus \{b\}$. These maps yield a commutative diagram

$$\begin{array}{ccccc}
H_{m+1}(X_{m+1}^k, X_m^k) & \longrightarrow & H_m(X_m^k) & \longrightarrow & H_m(X_m^k, X_{m-1}^k) \\
\downarrow & & \downarrow & & \downarrow \\
H_{m+1}(X_{m+1}^{k+1}, X_m^{k+1}) & \longrightarrow & H_m(X_m^{k+1}) & \longrightarrow & H_m(X_m^{k+1}, X_{m-1}^{k+1})
\end{array}$$

whose left and right vertical arrows are injective chain maps

$$C_{m+1}^k \hookrightarrow C_{m+1}^{k+1}, \quad C_m^k \hookrightarrow C_m^{k+1}$$

and whose composed horizontal maps are the boundary operators ∂^k_{m+1} resp. ∂^{k+1}_{m+1} of the chain complexes $\mathcal{C}^k$ resp. $\mathcal{C}^{k+1}$. Hence, we have an inclusion map $\mathcal{C}^k \hookrightarrow \mathcal{C}^{k+1}$. Such satisfies the relations

$$\ker \partial^k_m = \ker \partial^{k+1}_m \cap C^k_m$$
$$\operatorname{im} \partial^k_{m+1} = \operatorname{im} \partial^{k+1}_{m+1} \cap C^k_m$$

which imply the inclusions

$$H_m(\mathcal{C}^k, \mathbb{Z}) \subseteq H_m(\mathcal{C}^{k+1}, \mathbb{Z})$$

for $m \leq \ell + 1$. For $m > \ell + 1$ it holds trivially true that

$$H_m(\mathcal{C}^k, \mathbb{Z}) = H_m(\mathcal{C}^{k+1}, \mathbb{Z}).$$

In order to conclude, we need to prove that for all m

$$H_m(\mathcal{C}^k, \mathbb{Z}) \subseteq H^{\mathrm{top}}_m(\mathscr{C}, \mathbb{Z}).$$

Observe that for $m \leq k$ there is an inclusion $X_m \subseteq X^k_m$, whereas for $m > k$ we have $X_m = X^k_m$. The asserted inclusion of homology groups now follows from Theorem 8.11 and $H^{\mathrm{top}}_m(\mathscr{C}^k, \mathbb{Z}) = H^{\mathrm{top}}_m(\mathscr{C}, \mathbb{Z})$ which holds true, because

$$H^{\mathrm{top}}_m(\mathscr{C}^k, \mathbb{Z}) = H_m(X^k_\ell) = H_m(X_\ell) = H^{\mathrm{top}}_m(\mathscr{C}, \mathbb{Z})$$

for any $\ell > m$. □

Corollary 8.17. *Algorithm 8.16 computes $b^{\mathrm{top}}_i(\mathscr{C}(\mathcal{X}))$.*

Remark 8.18. *Each successive intermediate step in Algorithm 8.16 is performed by an Euler-Poincaré operator which simultaneaously increases the set B and the Betti numbers. It terminates when all Betti numbers equal the topological ones.*

8.4. **Examples.** The following examples illustrate some instances of Algorithm 8.16 which can be treated in an explicit way.

Example 8.19. *An important intermediate step in Algorithm 8.16 occurs when the boundary of a region is empty or connected.*

In the first case, there is a a terminal node in $\Gamma(\mathscr{C})$ of the following form:

$$\xrightarrow{\ \alpha\ } \overset{i}{\bullet}$$

where $i > 0$.

Case $\alpha \neq 0$. *If the weight α is nonzero, then adjoin an edge as such:*

$$\xrightarrow{\ \alpha\ } \overset{i}{\bullet} \xrightarrow{\ 0\ } \overset{0}{\bullet}$$

This is effected by the Euler-Poincaré operator $X_0 + Y_0$.

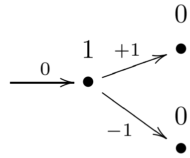
Case $\alpha = 0$. *If $\alpha = 0$, we distinguish two subcases:*

$i > 1$. *Then extend to*

$$\xrightarrow{\ 0\ } \overset{i}{\bullet} \xrightarrow{\ 1\ } \overset{i-1}{\bullet} \xrightarrow{\ 0\ } \overset{0}{\bullet}$$

using the Euler-Poincaré operator $(X_0 + Y_0) + (X_{i-1} - Y_i)$: the extra node yields X_0, together with the right arrow of weight zero yields $-Y_0$; the middle node yields X_{i-1}; the middle arrow of non-zero weight yields $-Y_i$. Note that there is no change *in b_{i-1}: the increase given by the new outgoing arrow of zero weight is compensated by the contribution of the new incoming arrow of non-zero weight.*

$i = 1$. *Then the edge is completed with two new vertices to*



This uses the Euler-Poincaré operator $(X_0+Y_0)+(X_0-Y_1)$, as can be seen similarly as in the previous subcase.

As to the case of disconnected boundary, we assume now that each component has undergone the previous process. For $b \in B$, we denote by Γ_b the architectural complex obtained from $\mathscr{C}$ by deleting b and all regions not adjacent to b in $\Gamma(\mathscr{C})$.

For each $b \in B_2$ having the property $b_0(\Gamma_b) > 1$ connect a fixed component C to any other component C' of Γ_b in the following way: let $v, v' \in B_0$ be terminal nodes of C and C', respectively. Now, a new edge c is introduced to B_1 whose boundary is precisely $v - v'$. This procedure is illustrated in Figure 9. This is effected by the Euler-Poincaré operator $X_1 - Y_0$. After completion, the new Betti number $b_0(\Gamma_b)$ is 1.

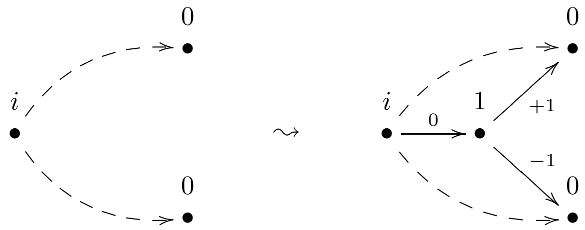


FIGURE 9. Connecting the boundary.

Next, apply the same procedure for all $b \in B_i$ $(i > 2)$ such that Γ_b is not connected. Complete the procedure first for n fixed, then increase n by one. This is also effected by $X_1 - Y_0$.

Remark 8.20. *The operators used in Example 8.19 are indeed Euler-Poincaré operators. In order to check this we need only verify the operator $X_{i-1} - Y_i$ for $i > 1$. But this one can be written as:*

$$X_{i-1} - Y_i = X^{i-1} - X^{\dim \mathscr{C}+1+i} = X^{i-1}(1 - X^{\dim \mathscr{C}+2})$$

which is indeed an Euler-Poincaré operator.

Example 8.21. *Figure 10 illustrates the transformation of architectural complexs to a cw-complex. The Betti numbers of the left complex in the top are readily computed: $b_0 = 0$ because there are no 0-regions; $b_1 = 1$ because there are two 1-regions* Circle, Line *with zero boundary, and the boundary of the 2-region* Area *is (with some choices of orientation)*

$$\partial(\text{Area}) = 1 \cdot \text{Circle} + 0 \cdot \text{Line},$$

which together yield the first Betti number. The resulting cw-complex has the expected Betti numbers. Note that the first Euler-Poincaré operator is

$$2(X_0 + Y_0) + (X_0 - Y_1) = 3X_0 + 2Y_0 - Y_1.$$

The topological Betti numbers of the ring are what they should be: $b_0 = b_1 = 1$.

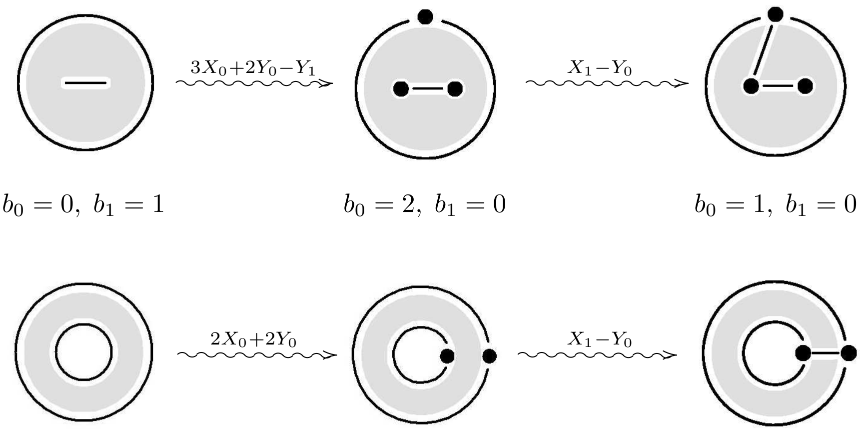


FIGURE 10. From architectural complex to cw-complex.

The last example shows the motivation coming from architecture.

Example 8.22. *Consider the floorplan as in Figure 11. Its corresponding archti-*

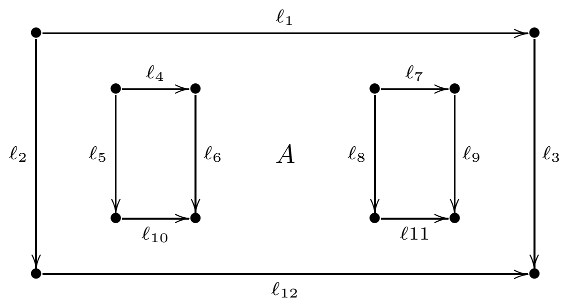


FIGURE 11. A floorplan with two inner courts.

tectural complex $\mathscr{C}$ *consists of the* 2*-region* A*, the lines* $\ell_1, \dots, \ell_{12}$ *and* 8 *points. The boundary of* A *is*

$$\begin{aligned} \partial A = & -\ell_1 + \ell_2 - \ell_3 + \ell_{12} \\ & + \ell_4 - \ell_5 + \ell_6 - \ell_{10} \\ & + \ell_7 - \ell_8 + \ell_9 - \ell_{11} \end{aligned}$$

Here, the topological boundary of A *has three connected components. An instance of Algorithm 8.16 then yields the cw-complex illustrated in Figure 12.*

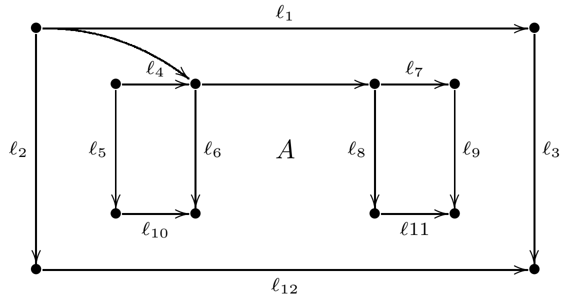


FIGURE 12. A floorplan with two inner courts.

9. Conclusion

Euler-Poincaré operators are defined as a generalisation to arbitrary dimension of the traditional Euler-operators from volume modeling. They originate in the Euler-Poincaré formula for chain complexes. Such an operator is realised as a class of operations on a given complex X which consistently results in another complex. If X is a cw-complex, then the result is not always again a cw-complex. For applications in architectural modeling, the notion of cw-complex is too restrictive, as e.g. building elements may have holes. Therefore, we introduce the notion of architectural complex via partial linear algebra and obtain in this way a topological interpretation of a chain complex whose boundary operator has most zeros removed. The remaining zeros allow the precise location of loops or shells of any dimension within the topological model.

There are infinitely many Euler-Poincaré operators. However, we show that they all can be generated by a finite set of so-called elementary Euler-Poincaré operators. These, in turn, can be built up from two "fundamental" operators through elementary algebraic operations. In fact, this is effected by polynomial multiplication. The interpretation of Euler-Poincaré operators allows to give explicit formulae for the expansion of a given operator into the elementary operators. The fact whether the dimension is even or odd makes a difference, and we take care of this. We believe that in principle, the formulae could be derived also using the Gauss elimination method for systems of linear equations, but using polynomials seems to us more elegant for this scope.

The main problem of architectural complexes is that, in contrast to cw-complexes, the Betti numbers do not in general allow the interpretation as numbers of connected components, loops or holes of any dimension, of the corresponding topological realisation. This deficiency can be overcome for those architectural complexes obtained by glueing certain types of manifolds in a similar manner as in the construction of cw-complexes. Such an architectural complex can be transfored into a cw-complex $\mathrm{cw}(X)$ by Euler-Poincaré operators which do not change the topological Betti numbers. These are then correctly given as the Betti numbers of $\mathrm{cw}(X)$.

Let us finally remark for future work that architectural application demands a relative version of the theory of Euler-Poincaré operators, because the making of details can be described by complex morphisms, or topologically: by continuous maps.

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