Topological Houses

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Abstract

The concept of a “house” is formalised in such a way that its topological properties can be encoded in a relational database without loss of information in many important cases.

1 Introduction

Many properties of houses are of topological nature. This is why an encoding of houses in a database which can handle their topologies is very useful and desired. E.g. DIME is a first step towards this in the two-dimensional case: it can deal with orientations of embedded planar graphs, however it looses some important topological information. The problem of three-dimensional encoding is solved here for a large class of houses by first giving an axiomatic description of a simplified concept of ”house” as a certain generalisation of a cw-complex and, secondly, by generalising local observation structures of embedded unconnected planar graphs discussed in [Hid] to the three-dimensional case and proving that they allow retrieving the topological properties of these houses. Finally, a lossless representation of observation structures in a relational database structure which we call PLAV is given.

2 Definiton of Topological Houses

In order to be able to encode the topological properties of something like a house into a database, we must formalise the definition of a house and its topological properties.

Definition 2.1. *A* topological house *is a compact, connected threedimensional subset H of $\mathbb{R}^3$, which is the union of finitely many cells satisfying the following conditions:*

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1. *The* 1*-skeleton of* H *is a graph, and any of its minimal loops is the boundary of a* 2*-cell of* H*, if the corresponding edges all lie in the closure of a* 2*-cell.*

2. *Each* 0*-cell lies in the closure of a* 2*-cell.*

3. *Each* 1*- or* 2*-cell lies in the closure of a* 3*-cell.*

4. *Any pair of* n*-cells (for fixed* $n \neq 2$*) is disjoint.*

5. *If* h_1 *and* h_2 *are* 2*-cells, then either* $h_1 \cap h_2 = \emptyset$*,* $h_1 \subseteq h_2$ *or* $h_2 \subseteq h_1$*.*

The 2*-cells of a topological house* H *are called* generalised walls*, and the* 3*-cells are called* rooms*. A cell inside another cell is called an* interior cell*.*

The simplest kind of house one can imagine is a threedimensional polyhedron. However, it has no doors, no windows, no stairs, and no conducts. Definition 2.1 takes all this into account. The 1-skeleton, for example, is in general an unconnected graph: one connected component is normally the skeleton of the actual building, whereas the other components are columns, pipes, ducts etc.

A generalised wall can be for example the interior of a wall or door, an opening in the floor, or (in any case) just the space filling out a loop in the 1-skeleton.

In order to tell whether two topological houses are considered to be of the same kind, we shall introduce the appropriate type of mappings between houses.

Definition 2.2. *Let* H *and* H' *be topological houses. A* house map $f\colon H \to H'$ *is a continuous map* $H \to H'$ *such that for every* $n \in \mathbb{N}$ *the image of an* n*-cell of* H *is an* n*-cell of* H'*. Houses* H *and* H' *are* equivalent *if there exist house maps* $f\colon H \to H'$ *and* $g\colon H' \to H$ *which are inverse to each other, i.e. with* $f \circ g = \mathrm{id}_{H'}$ *and* $g \circ f = \mathrm{id}_H$*. In this case, we shall write* $H \cong H'$*.*

3 Encoding Simple Houses

Generalising the PLA-structure of spatial databases for plane graphs discussed in [Hid], we define an observation structure on a topological house H.

Let H_n be the set of n-cells of H, and $D(H)$ a database structure on H. This is just a list of names for each n-cell of H. We assume that the name indicates also the dimension n of the corresponding cell. Also, we take up an extra "cell": the complementary $H_\infty := \mathbb{R}^3 \setminus H$, called the *outside* of the house. By abuse of language, we pretend as if H_∞ were a cell of H.

Take a point $p \in H_0$, and draw a small enough 2-sphere S_p around p. The intersection of S_p with an n-cell of H will either be empty or an $(n-1)$-cell.

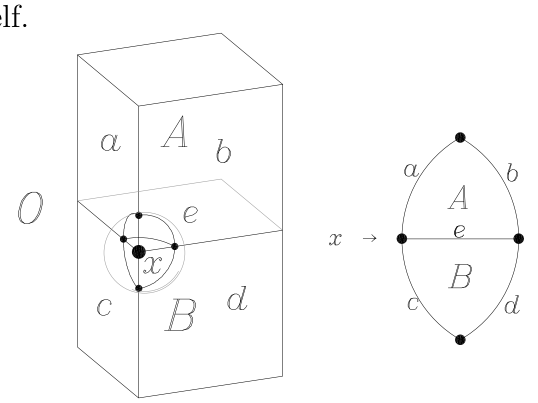
This induces a graph on S_p — a connected planar graph Γ_p. We convene to put the point ∞_p at infinity on each S_p in the outside of the house, if $S_p \cap H_\infty \neq \emptyset$. Otherwise, ∞_p will be an arbitrary point in an arbitrary face of the graph. "Small enough" means that the induced graph on any smaller sphere will be the same as Γ_p, if we do not care about the lengths of edges.

Let $D_H(\Gamma_p)$ be the induced database for Γ_p as an embedded plane graph: the name of an n-piece $\gamma \in \Gamma_p$ (i.e. a point, a line or an area) is given by a unique number and the name of the smallest $(n+1)$-cell in H containing γ.

Definition 3.1. *An* observation structure *in $p \in H_0$ is a triple $Obs_H(p) := (p, \Gamma_p, PLA(p))$, where $PLA(p)$ is a PLA-structure on $D_H(\Gamma_p)$.*

An observation structure *on H is a family $Obs_H := (Obs_H(p))_{p \in H_0}$ of observation structures in all points p.*

For the convenience of the reader, we recall from [Hid] the definition of a PLA-structure on $D_H(\Gamma_p)$: it is a tuple (P, L, A, Obs, ∞_p), where P, L and A are the sets of names for points, lines and areas of Γ_p, respectively; and Obs is a function which maps every name in P to a circular list of the pieces obtained by intersecting a small circle around each vertex of Γ_p with the plane graph Γ_p itself.



A local observation structure in the vertex x given by a graph on a small sphere around x

The importance of the reference cell H_∞ will become evident when trying to recover houses from observation structures.

Definition 3.2. *A topological house H is called* simple*, if every edge of its 1-skeleton lies in the boundary of a room of H.*

In other words, there are no columns allowed, and all pipes are integrated in the walls.

Remark 3.3. *The advantage of considering a simple house H instead of an arbitrary house is that it corresponds to a partitioning of $\mathbb{R}^3$ which gives rise to a more simple m-complex $\mathfrak{X}$: the 0-, 1- and 3-dimensional pieces of $\mathfrak{X}$ are in fact cells, whereas the 2-dimensional pieces are connected manifolds obtained by taking any maximal 2-cell X of H and cutting out all 2-dimensional holes defined by the interior cells of X. $\mathfrak{X}$ is called the* architectural complex *of H. An* *architectural complex can easily be made into a cw-complex $\mathfrak{X}_{cw}$ by connecting all components of the 1-skeleton by non-intersecting paths through the closures of the 2-pieces containing them (see also [P]).*

Theorem 1. *Given an observation structure Obs_H on a simple house H, all houses that can be constructed from it (and that have the same observation structure) are equivalent to H.*

Proof. Let F^n denote the n-skeleton of a complex F.

For each p, the graph Γ_p together with $D_H(\Gamma_p)$ gives us for each $n \geq 1$ local pieces

$$H_p^n := \{p\} \cup \bigcup_{\gamma \in \Gamma_p^{n-1}} C_\gamma,$$

where C_γ is the minimal cell of H containing γ. It is clear that the family of all $\{H_p^n\}_{p \in H_0}$ is an open covering of H_n. The observation structure $Obs_H(p)$ gives us an embedding

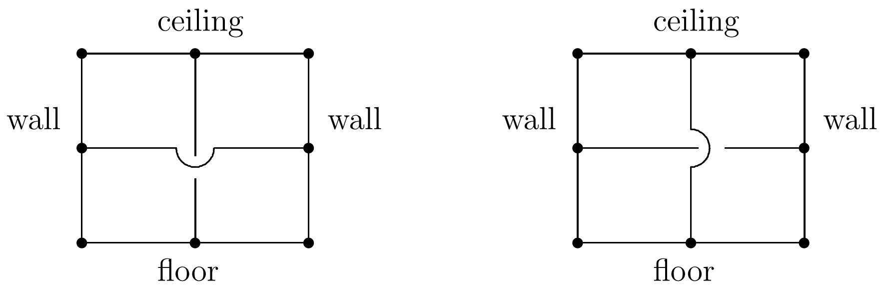
$$j_p^n \colon H_p^n \to \mathbb{R}^3$$

for $n = 3$ which, in turn, induces embeddings j_p^n for all smaller n. These embeddings can be pasted together along induced embeddings

$$j_{pq}^n \colon H_p^n \cap H_q^n \to \mathbb{R}^3$$

to an embedding $j^n \colon \tilde{H}^n \to \mathbb{R}^3$. The m-complex obtained by cutting out holes as in Remark 3.3 is the architectural complex of a simple house $\tilde{H}$ embedded into $\mathbb{R}^3$ and equivalent to H, as all its n-skeletons $\tilde{H}_n$ are homeomorphic to the H_n by the uniqueness property of pasting. □

The condition in the definition of "simple" cannot be removed, as the following example shows:



If both houses have only one room, then they are equivalent, as can be seen by rotating around the vertical pipe through the room. But in the case of more rooms, the houses are in general not equivalent, and therefore observation structures cannot distinguish between the two houses.

4 PLA-Structures in Relational Databases

Given an embedded planar graph Γ, let the embedding into the plane be encoded by its PLA-structure $(P, L, A, Obs, a^{\infty})$. In order for this structure to fit into a relational database, the circular lists in Obs must be represented as relations in first normal form (NF1). If $\langle a_0, l_0, a_1, l_1, \ldots, a_n, l_n \rangle$ is the observation of the point $p_0 \in P$, then an encoding like

$$Obs(p_0) = \{(0, a_0), (1, l_0), \ldots, (2n, a_n), (2n+1, l_n)\} \subset \mathbb{N} \times (L \amalg A)$$

would be straightforward, but there is a more efficient way.

Let $e \colon L \to P \times P$ be an orientation on L making Γ into an oriented graph $\Gamma_P := (P, L, e)$. Also, let $\Gamma_A := (A, L, f)$ be the dual graph of Γ_P with respect to the planar embedding whose orientation $f \colon L \to A \times A,\ l \mapsto (o_f(l), t_f(l))$ is defined such that for each $l \in L$ the origin $o_f(l)$ is the area to the left of l and the target $t_f(l)$ is the area to the right. Γ_A is called the *area adjacency graph* (AAG) of the embedding. With the map

$$\Phi \colon L \mapsto (P \times P) \times (A \times A),\ l \mapsto (e(l), f(l)),$$

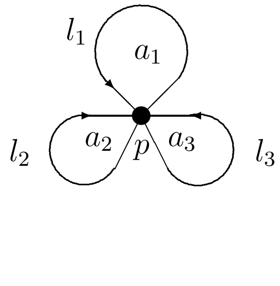
the quintuple $(P, L, A, \Phi, a_{\infty})$ is known as DIME (Dual Independent Map Encoding). Usually, for each $l \in L$ the quaduple $\Phi(l)$ is given out.

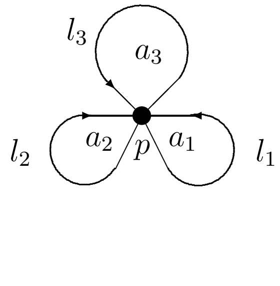
DIME is commonly used for encoding topological information in geographic information systems. Although an efficient data structure, it has the disadvantage of losing some topological information.

Consider a clover-like graph: one point, three lines, three 2-cells and one exterior area. It has the following PLA-structure:

$$\begin{aligned} (P = \{p\}, \quad & L = \{l_1, l_2, l_3\}, \quad A = \{a_1, a_2, a_3, a_{\infty}\}, \\ & Obs(p) = \langle a_{\infty}, l_1, a_1, l_1, a_{\infty}, l_2, a_2, l_2, a_{\infty}, l_3, a_3, l_3 \rangle, \quad a_{\infty}). \end{aligned}$$

Let its DIME be $\begin{cases} l_1 \colon & (p, p, a_1, a_{\infty}) \\ l_2 \colon & (p, p, a_{\infty}, a_2) \\ l_3 \colon & (p, p, a_3, a_{\infty}) \end{cases}$. Every line $l \colon (p, q, a, b)$ gives the subsequence $\langle b, l, a \rangle$ in the observation structure of p and $\langle b, l, a \rangle$ in that of q. Unfortunately, this is not enough for recovering this example uniquely:





Only the first clover has the correct observation structure. Uniqueness is obtained by enumerating all incident lines in the order they appear locally at every given point p:

$$ord(p) := \bigcup_{i=1}^{m} \{(i, \varphi(l_{\sigma(i)}))\},$$

where

$$\varphi\colon L \to \{\pm 1\} \times L,\ l \mapsto \begin{cases} (+1, l), & \text{if } l \text{ comes into } p \\ (-1, l), & \text{if } l \text{ goes out of } p \end{cases}$$

and $l_{\sigma(i)}$ is the line on the i-th place in the list obtained from $Obs(p)$ by deleting all area names.

In this example, $Ord(p) = \{(1, l_1), (2, -l_1), (3, -l_2), (4, l_2), (5, l_3), (6, -l_3)\}$ for the left clover and $Ord(p) = \{(1, l_1), (2, -l_1), (3, l_3), (4, -l_3), (5, -l_2), (6, l_2)\}$ for the right clover.

In fact, this works for any connected graph embedded into a compact, orientable surface [GT]. And for the general case, the dual graph can be successfully used, as we will see now.

In what follows, $\mathfrak{R} = \mathfrak{R}(\underline{\pi_1}, \dots, \underline{\pi_s}, \pi_{s+1}, \dots, \pi_t)$ will denote any relational scheme with $\Pi = \{\pi_1, \dots, \pi_s\}$ as a key and some (more or less specified) additional functional dependencies.

Here, we consider

$$\begin{array}{lcl} Gr(\underline{\pi_L}, origin, target) & \subseteq & \mathcal{P}(L \times P \times P) \\ Aag(\underline{\pi_L}, left, right) & \subseteq & \mathcal{P}(L \times A \times A) \\ Ord(\underline{\pi_P}, \underline{\pi_{\mathbb{N}}}, dir, line) & \subseteq & \mathcal{P}(P \times \mathbb{N} \times \{\pm 1\} \times L) \end{array}$$

where Gr represents the graph, Aag the AAG, and Ord respecting the functional dependencies $\{\pi_P, \pi_{\mathbb{N}}\} \to \{dir, line\}$, represents ord from above. $\mathcal{P}(X)$ means the set of all subsets of X and π_X the canonical projection onto X (all other attributess are supposed to be self-explaining).

Note that the join of Aag and G yields DIME.

In our setup, Gr is actually redundant, as the following lemma shows.

Lemma 4.1. *The relational schemes Aag and Ord together are a lossless encoding of embedded planar graphs (up to topological equivalence).*

Proof. The PLA-structure of an embedded planar graph Γ can easily be retrieved from the directed dual graph and all $ord(p)$ with $p \in P$. By [Hid, Theorem 1] the PLA-structure recovers the embedding of Γ into the plane up to isotopy, which gives the result. □

A simple SQL query retrieves the observation structure $obs(p_0)$ of a given point $p_0 \in P$ in a form like:

$$\bigcup_{i=1}^{m} \{(i, l_{\sigma(i)}, a_{\tau(i)})\} \subseteq \mathbb{N} \times L \times (A \cup \{a_\infty\}),$$

a relation in NF1. So this data structure is also a lossless representation of topological information.

5 PLAV-Structures in Relational Databases

Here, we extend the relational PLA-structure from the preceding section to three dimensions.

For a topological house H let $\mathcal{X}$ be its architectural complex with points P, lines L, areas A and volumes V. Orient the lines L and the areas A.

Since the intersection of a volume or an area with a local sphere S_p is in general unconnected, some extra book keeping is needed. The relational schemes

$$\begin{aligned} locL(\pi_P, \pi_{\mathbb{N}}, line) &\subseteq \mathcal{P}(P \times \mathbb{N} \times L) \\ locA(\underline{\pi_P}, \underline{\pi_{\mathbb{N}}}, area) &\subseteq \mathcal{P}(P \times \mathbb{N} \times A) \\ locV(\underline{\pi_P}, \underline{\pi_{\mathbb{N}}}, volume) &\subseteq \mathcal{P}(P \times \mathbb{N} \times V) \end{aligned}$$

enumerate lines, areas, resp. volumes locally as points, lines, resp. areas on each S_p.

Local parts of the VAG on the spheres S_p are obtained by

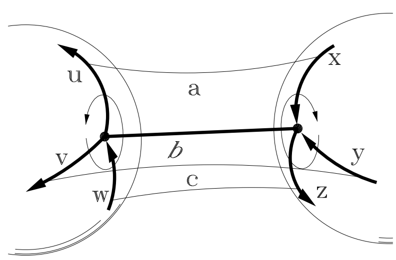
$$locVag(\underline{\pi_P}, \underline{\pi_{\mathbb{N}}}, left, right) \subseteq \mathcal{P}(P \times locA.\pi_{\mathbb{N}} \times locV.\pi_{\mathbb{N}} \times locV.\pi_{\mathbb{N}}),$$

where for $a \in A$ a connected component of $a \cap S_p$ is taken together with its left and right neighbouring component of $v \cap S_p$ for some $v \in V$.

The pasting (in the case of simple houses) is encoded in

$$glue(\underline{line}, \underline{\pi_{\mathbb{N}}}, odir, oarea, tarea) \subseteq \mathcal{P}(L \times \mathbb{N} \times \{\pm 1\} \times A \times A).$$

Here, a circular list of observed areas $a \in A$ incident in a line $l \in L$ is encoded in this way: if the i-th observation of an area is a, then $odir$ is the local direction of the corresponding component e of $a \cap S_{o(l)}$, $oarea$ is its local name and $tarea$ the local name of the component $\tilde{e}$ of $a \cap S_{t(l)}$ corresponding to this i-th observation (the direction of $tarea$ is the inverse of $odir$).



An example of a local observation of a line whose $glue$ is

$$\begin{aligned} &(l, 1, -1, u, x) \\ &(l, 2, -1, v, y) \\ &(l, 3, +1, w, z) \end{aligned}$$

Consider the system of relational schemes

$$\text{PLAV}: \quad \{\mathfrak{X}_1, locL, locA, locV, locVag, glue\},$$

where $\mathfrak{X}_1(\underline{line}, origin, target) \subseteq \mathcal{P}(L \times P \times P)$ represents the 1-skeleton $\mathcal{X}_1$ oriented as L. This is the database version of Obs_H from Section 3.

Theorem 2. *The system of relational schemes* PLAV *is a lossless representation (up to equivalence) of local point observations of houses in a relational database.*

Proof. From PLAV, one easily obtains the local versions of Aag and Ord for each local graph $\Gamma_{o(l)}$ on $S_{o(l)}$ with $l \in L$, whereas in $t(l)$, the observation list has to be reversed. So, by Lemma 4.1, all Γ_p are recovered. □

By construction, we get together with Theorem 1:

Corollary 5.1. *PLAV is a lossless encoding of simple houses.*

6 Conclusion

A relational data structure for encoding the PLAV-structure of houses using local PLA-structures was obtained by exploiting the combinatorics of their architectural complexes. Possibly further normalisation can be done, making PLAV an interesting approach towards encoding three-dimensional topological information in databases.

The loss of information observed in general is likely to be covered by some knot theory, or by using the metric of the ambient space as do geographic information systems which use DIME.

In a next step, we expect PLAV to be useful for encoding higher dimensional topological spaces, in particular (architectural) space-time complexes.

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