From image processing to topological modelling with p-adic numbers

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Abstract

Encoding the hierarchical structure of images by p-adic numbers allows for image processing and computer vision methods motivated from arithmetic physics. The p-adic Polyakov action leads to the p-adic diffusion equation in low level vision. Hierarchical segmentation provides another way of p-adic encoding. Then a topology on that finite set of p-adic numbers yields a hierarchy of topological models underlying the image. In the case of chain complexes, the chain maps yield conditions for the existence of a hierarchy, and these can be expressed in terms of p-adic integrals. Such a chain complex hierarchy is a special case of a persistence complex from computational topology, where it is used for computing topological barcodes for shapes. The approach is motivated by the observation that using p-adic numbers often leads to more efficient algorithms than their real or complex counterparts.

Keywords: p-adic numbers, scale space, segmentation, algebraic topology

1 Introduction

Low level vision refers to the estimation of the scene underlying a given image. Here, the dynamics in the geometry of a single image has become the main concern. The usual method is to filter the image iteratively, which produces a one-parameter family of images, starting with the original. This family is termed *scale space*. The idea is that boundaries between objects should survive as long as possible within the scale space, while homogeneous regions should become flattened more rapidly. This process is described by a diffusion equation $\partial_t X = \mathcal{D}X$, where $\mathcal{D}$ is a local differential operator acting on the image $X = X(\Sigma, t)$ with Σ the image domain. The variable t parametrises the different filtering stages.

A very general framework for low level vision is provided by ideas from high-energy physics. Namely, if Σ is viewed as the "worldsheet", and if the target space is allowed to be any Riemannian manifold, then X describes a "string" whose dynamic is governed by the so-called *Polyakov action*. It provides a measure on these maps X. This allowed [19] to unify many seemingly unrelated scale space methods and to provide new and improved ways of smoothing and denoising images.

In computer vision, the features detection is one step towards an automated vision system. Another step is to match features in two images in order to be able to understand the 3D scene and dynamics. This can be done by estimating the camera motion from two views, and methods from projective and algebraic geometry enter the scene at an early stage, as for example in [20]. The beginning of this present century witnesses the application of sophisticated methods from computational commutative algebra in order to rephrase the equations into a form from which solutions can be obtained with relative ease. The relationship between the views is established by finding correspondences between point pairs taken from both images. The fundamental matrix faithfully encodes the geometric relationship between the two images. For normalised cameras, the fundamental matrix coincides with the essential matrix. In general, the two matrices are related through the camera calibration. Hence, if the calibration is known, it is sufficient to estimate the essential matrix in order to solve the relative pose problem. From a conceptional as well as a computational point of view, it makes sense to use only few correspondences of image points in order to estimate the essential matrix E. And different samples of n correspondences lead to a set of candidate essential matrices from which an optimal choice can be made. This method is called RANSAC: Random Sample Consensus.

Segmentation often aims at understanding single images. Here, the information on the pixels are classified in order to find contiguous regions which are sufficiently homogeneous. This can be used for object recognition or tracking through image sequences. Another objective is that of constructing from images of some scene a model. Cartographic maps are often 2D-models, possibly toghether with a height relief. Building or city models provide 3D information. However, any finite or infinite dimensional model is conceivable, as e.g. time or other attributes each can provide for an extra dimension in the model, or the data can be transformed into some space of functions where the modelling is to take place. Very often, a *topological* model is of interest in order to be able to understand properties which are independent of the geometry, like connectivity, adjacency or the number of minimal paths, loops or holes. The hierarchical approach leads to multi-representations, which in our case are hierarchical topological models. In computational topology, this has lead to the use of a family of chain complexes connected by chain maps, called a *persistence complex*, in order to derive an algorithm for computing persistence barcodes, a homological invariant for shape data [8].

According to Murtagh [16], ultrametricity is pervasive in observational data, and this offers computational advantages and a well understood basis for developping data processing tools originating in p-adic arithmetic. Consequently, p-adic data encoding becomes necessary. In [3] it has been shown that the choice of the prime number p is arbitrary, if one takes an unramified extension field K of $\mathbb{Q}_p$ of sufficient degree. The reason is that K has a *Bruhat-Tits tree* T_K, a $(q+1)$-regular infinite tree with q increasing with the degree of K, and at whose boundary lie the points of K. The p-adic encoding simply identifies the data with some boundary points of that tree, and this defines the tree of hierarchies within the data. In other words $p = 2$ can be taken, which is usually the computationally most advantageous prime number. In particular, the p-adic Newton iteration method, known in number theory as Hensel's lemma, is most efficient for $p = 2$.

In this article, we are interested in two aspects from a p-adic point of view: low level vision and topological modelling from segmentation. Both aspects ask for p-adically encoded images. This means that a p-adic (gray-scale) image is defined as a function $\Sigma \to \mathbb{R}$, where the image manifold Σ is a p-adic space. First, hierarchical image encoding methods based on interval subdivision are reviewed. In our case, we can use $p = 2$ and represent image points in several ways by pairs of binary expansions

$$a = \sum_{n=0}^{\infty} a_i 2^i$$

with coefficients a_i equal to 0 or 1. These expansions can be infinite, theoretically. Practically, the finiteness of resolution means approximation through truncation. The framework for this method is p-adic geometry which has been applied in video segmentation and data analysis [2, 16, 3]. A hierarchical segmentation in which segments at one level are subdivided in the next level provides another way of p-adic image encoding. This time the field of definition can be an extension field of higher degree, depending on the maximal number of children vertices in the tree.

We will review in Section 3 some natural 2-adic encodings which allow to view gray-scale images as real-valued functions on p-adic spaces. This should be understood as an invitation to develop image processing methods originating in p-adic functional analysis.

In Section 4, we formulate a p-adic scale space equation from the Polyakov action on the Bruhat-Tits tree. This action was introduced to p-adic string theory by [22]. The result is a p-adic diffusion equation as the counterpart to the diffusion equation obtained for scale spaces of images with real coordinates.

Section 5 describes how a topological structure on the finite p-adic code leads to a hierarchy of compatible topological structures on each level. Important is the condition that segments are open. Then, in the case of chain complexes, conditions are derived for the system of boundary maps in order to obtain cellular maps between the levels. These conditions are also given in terms of p-adic integrals.

The article is preceeded by a short Section 2 on p-adic numbers.

2 p-adic numbers

Kurt Hensel's important contribution to number theory was to view numbers as analytic functions on some imagined "Riemann surface". In this imaginary situation, the "places" are given by the prime numbers p which play the role of a local coordinate[1], and then the number n has "locally" a unique power series expansion

$$n = \sum_{\nu=0}^{\infty} n_\nu p^\nu,$$

which in the case of natural numbers n is in fact a finite expansion with coefficients $n_\nu \in \{0, \dots, p-1\}$. The p-adic metric is given by the length of the common initial part:

$$|n - m|_p = p^{-\nu}, \tag{1}$$

if $m = n_0 + \dots + n_{\nu-1}p^{\nu-1} + m_\nu p^\nu + \dots$ and $m_\nu \neq n_\nu$. This is an ultrametric, i.e. the strict triangle inequality

$$|x + y|_p \leq \max\left\{|x|_p, |y|_p\right\}$$

holds true. Allowing infinite expansions (1) means completion with respect to the p-adic metric, and the completed space $\mathbb{Z}_p$ of *p-adic integers* contains the usual integers $\mathbb{Z}$ as a dense subset. Examples of negative numbers are

$$\sum_{\nu=0}^{\infty} p^\nu = \frac{1}{1-p}, \qquad \sum_{\nu=0}^{\infty} (p-1)p^\nu = -1$$

The primality of p guarantees that there are no zero-divisors in $\mathbb{Z}_p$, and the field of fractions $\mathbb{Q}_p$ can be formed which densely contains the rational numbers $\mathbb{Q}$. Just like in the function-theoretic case, the p-adic numbers thus correspond to the meromorphic functions:

$$\mathbb{Q}_p = \left\{ \sum_{\nu=-N}^{\infty} x_\nu p^\nu \mid x_\nu \in \{0, \dots, p-1\} \right\}$$

and have a "Laurent series" expansion. Observe further that $\mathbb{Z}_p$ is the p-adic unit disk:

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p \mid |x|_p \leq 1 \right\},$$

and we have in $\mathbb{Q}_p$ an ultrametric space on which calculus can be performed.

p-adic approximation is given by finite expansions: $x = x_0 + \dots x_{n-1}p^{n-1} +$ higher order terms. That cut-off can be written by a congruence

$$x \equiv x_0 + \dots + x_{n-1}p^{n-1} \mod p^n, \tag{2}$$

from wich it follows that the p-adic expansion of x is given by an infinite sequence of congruences (2) with $n = 1, 2, 3, \dots$. And indeed,

$$\left| x - \sum_{\nu=0}^{n-1} x_\nu p^\nu \right|_p \leq p^{-n},$$

[1] In fact, this dream became true thanks to Grothendieck's concept of *scheme*: The "Riemann surface" is the affine scheme $\operatorname{Spec} \mathbb{Z}$, the space whose points are the prime ideals $p\mathbb{Z}$ for $p = 0$ or a prime number. Cf. e.g. [12].

we have convergence of these finite expansions to x for $n \to \infty$.

The ultrametric (1) reveals the tree-like structure of $\mathbb{Q}_p$. In fact, by taking as vertices the disks, and as edges the non-trivial inclusions of disks $A \subset B$ not having any other disk C strictly in between: $A \subset C \subset B$, one obtains the *Bruhat-Tits tree* T_p. It is a $p+1$-regular infinite tree, and its boundary is in one-one correspondence with $\mathbb{P}^1(\mathbb{Q}_p) = \mathbb{Q}_p \cup \{\infty\}$. Notice that the local structure of T_p means that the edges attached to each vertex is in one-one correspondence with $\mathbb{P}^1(\mathbb{F}_p)$, where $\mathbb{F}_p$ is the *residue field* $\mathbb{Z}_p/p\mathbb{Z}_p$ of $\mathbb{Q}_p$, the finite field with p elements. And indeed, the children of a vertex (= disk B) correspond uniquely to the residue classes modulo p of the unit disk, after translating B into the unit disk and rescaling to $B' = \mathbb{Z}_p$.

Let us remark that $\mathbb{Q}_p$ is endowed with a Haar measure dx such that

$$\int\limits_{\mathbb{Z}_p} dx = 1,$$

i.e. the unit disk has volume 1. In particular, disks have volume equal to their diameter. An important example integral is

$$\int\limits_{\mathbb{Z}_p} dx \, |x|_p^s = \sum_{\nu=0}^{\infty} (p-1)p^{-\nu s}p^{-(\nu+1)} = \frac{p-1}{p} \cdot \frac{1}{1-p^{-s-1}}$$

for $\mathrm{Re}(s) > 1$. This follows from integrating the locally constant $|x|_p^s$ on spheres $|x|_p = p^{-\nu}$ which consist of $p-1$ balls of volume $p^{-\nu-1}$, and then summing up the geometric series.

Finite field extensions K of $\mathbb{Q}_p$ play an important role in p-adic theories. They have a Bruhat-Tits tree T_K which is $q+1$-regular, where q is the cardinality of the residue field $\mathbb{F}_{p^f}$ of K. There is again a Haar measure dx on K. For $n = \dim_{\mathbb{Q}_p}(K)$, there is the relation $n = ef$. If $e = 1$, then K is *unramified*, otherwise *ramified* over $\mathbb{Q}_p$. The unramified case has been used in [4] in the context of classification. This allows for dendrograms with arbitrary branching without having to change the prime number p. In particular $p = 2$ is sufficient, and this small prime number yields the most efficient algorithms in general. The ramified field extensions have been of importance in string theory. Ghoshal [10] has used them for giving a meaning to the so-called "$p \to 1$ limit" by the sequence of lattice discretisations of the string world sheet for each $e \geq 1$. This, in turn, gave a physical meaning to p-adic string theory.

An Introduction to p-adic numbers can be found e.g. in [11]. A brief review of applications of p-adic numbers in physics and other sciences is presented in [9].

3 p-adic encoding of images

A 2-adic encoding of square $2^N \times 2^N$-images can be obtained by a hierarchical subdivision as in Fig. 1. Essentially, there are two approaches for the encoding.

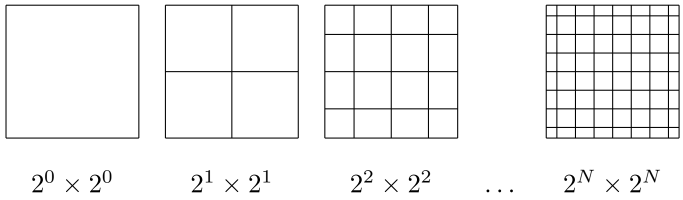

Figure 1: Hierarchical subdivision of an image

In the *bottom-up encoding*, the squares at highest resolution are assigned to level N, with decreasing level at higher hierarchy, level 0 representing the full image cluster. The encoding scheme for the x-coordinate is to traverse a path from bottom to top, and collect a coefficient $a_\nu = 0$ for each right turn, and $a_\nu = 1$ for left turns. This yields the expansion

$$x = \sum_{\nu=0}^{N} a_\nu 2^{-\nu}.$$

Fig. 2 (left) exemplifies this with

$$x_1 = 0, \qquad x_2 = 2^{-1}, \qquad x_3 = 2^{-2}, \qquad x_4 = 2^{-2} + 2^{-1}.$$

The intensities (gray values) on the image grid can be viewed as *locally constant* functions $f\colon \mathbb{Q}_p \to \mathbb{R}$, since vertices in the dendrogram at level ν can be viewed as p-adic disks of radius p^ν. With the bottom-up encoding the functions are constant on all translates of the unit disk $\mathbb{Z}_p$, and methods from p-adic functional analysis are ready for application. The functions which are constant on the sets $x + \mathbb{Z}_p$ are in one-one correspondence with functions on the co-set space $\mathbb{Q}_p/\mathbb{Z}_p$. Different bottom-up encoding approaches have been employed by Murtagh in classification [17]. The following section outlines an application to low level vision.

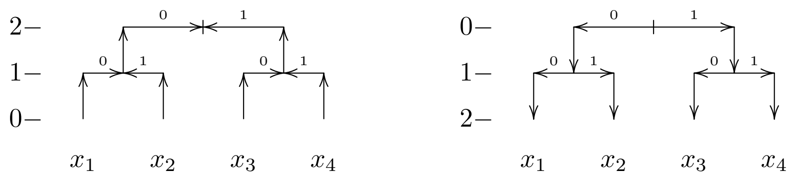

Figure 2: Left: bottom-up encoding, right: top-down encoding of a dendrogram

The *top-down encoding* reverses the order of bottom-up, and expansion is in positive powers of 2. This yields 2-adic integers for image coordinates, which turns out useful in [5], where the Hensel lemma can be applied. In Fig. 2 (right) one obtains

$$x_1 = 0, \qquad x_2 = 2^1, \qquad x_3 = 2^2, \qquad x_4 = 2^1 + 2^2.$$

In fact, this is the way we have constructed the Bruhat-Tits tree T_p in the previous section. A formal definition of the top-down encoding is given in [5]. Image-based p-adic encodings can be found in [13, 14].

4 Scale space through Polyakov action

A general framework for low level vision is proposed by Sochen et al. [19]. They consider low level vision as an input-output process given by a so-called *scale space equation*

$$\partial_t X = \mathcal{D} X,$$

where $\mathcal{D}$ is a local differential operator acting on the output vector $X(t)$ on the image, viewed as a manifold (Σ, g) with some metric g. In order to obtain a very general description, they take as X an embedding map

$$X\colon (\Sigma, g) \to (M, h)$$

into another surface (M, h). The *Polyakov action* is given by the functional

$$S[X, g, h] = \int_\Sigma d^m\sigma \, \sqrt{\det g}\, g^{\mu\nu} \partial_\mu X^i \partial_\nu X^j h_{ij}, \tag{3}$$

where $m = \dim \Sigma$, $g^{-1} = (g^{\mu\nu})$, and $\mu, \nu = 1, \ldots, \dim \Sigma$, $i, j = 1, \ldots, \dim M$ is summed over (Einstein's summation convention). And

$$\partial_\mu X^i := \frac{\partial X^i(\sigma^1, \sigma^2)}{\partial \sigma^\mu}$$

is the partial derivative. Polyakov writes down the action (3) in the case of a flat background space, i.e. $h_{ij} = \delta_{ij}$ [18].

The Euler-Lagrange equations yield

$$-\frac{1}{2\sqrt{\det g}} h^{i\ell} \frac{\delta S}{\delta X^\ell} = \frac{1}{\sqrt{\det g}} \partial_\mu \left(\sqrt{\deg g}\, g^{\mu\nu} \partial_\nu X^i \right) + \Gamma^i_{jk} \partial_\mu X^j \partial_\nu X^k g^{\mu\nu}, \tag{4}$$

where Γ^i_{jk} are the coefficients of the Levi-Cività connection. Sochen et al. propose to use the equation

$$\partial_t X^i = -\frac{1}{2\sqrt{\det g}} h^{i\ell} \frac{\delta S}{\delta X^\ell} =: \Delta_g X^\ell$$

as the scale space equation [19]. The operator Δ_g is the *Beltrami operator*, a natural generalisation of the Laplace operator to manifolds.

For example, the gray-scale image induces the embedding

$$X\colon \Sigma \to \mathbb{R}^3, \quad \sigma \mapsto (x, y, I),$$

and the metrics are $h_{ij} = \delta_{ij}$, and $g_{\mu\nu} = \delta_{\mu\nu}$. Then

$$S[X, g, h] = \int\limits_{\Sigma} d^2\sigma \, \left(|\nabla x|^2 + |\nabla y|^2 + |\nabla I|^2\right)$$

In the case that $x = \sigma^1, y = \sigma^2$, the usual heat operator acting on I is obtained by minimising the action S with respect to I. In other words, the usual scale space equation

$$\partial_t I = \Delta I,$$

is obtained from this action, as can be derived from the Euler-Lagrange equation (4).

In the p-adic case, we will not be quite as general as in the Archimedean case before. Our starting point is p-adic diffusion. This can be described as a symmetric jump process on $\mathbb{Q}_p/\mathbb{Z}_p$, with the transition probability P_{xy} depending only on their p-adic distance:

$$P_{xy} = e^{\rho(|x-y|_p)}.$$

The equation describing the evolution of probabilities is then given by

$$\frac{d}{dt} f(x,t) = \int\limits_{\mathbb{Q}_p} (f(y,t) - f(x,t)) P_{xy} \, dy, \tag{5}$$

where the right-hand side is a non-local pseudo-differential operator. Equation (5) has been successfully applied to replica symmetry breaking by Avetisov et al. [1].

Assume that the integral kernel is of the form

$$\rho(|x|_p) = C_p \, v_p(x)(1+\alpha) \ln p$$

with $\alpha > 0$, and $|x|_p = p^{-v_p(x)}$. This corresponds to activation barriers for the jump process having linear growth with respect to p-adic distance. In this case, equation (5) describes *p-adic Brownian motion.* For $-\log C_p = \frac{1-p^{-\alpha-1}}{1-p^{\alpha}} =: \Gamma_p(-\alpha)$, the role of the heat operator is played by the *Vladimirov operator*

$$D^\alpha f(x,t) = \frac{1}{\Gamma_p(-\alpha)} \int\limits_{\mathbb{Q}_p} \frac{f(x) - f(y)}{|x-y|_p^{1+\alpha}} \, dy, \tag{6}$$

where $\Gamma_p(\alpha)$ is known as the the p-adic *gamma function.* We propose that the diffusion equation

$$\frac{d}{dt} f(x,t) = -D^\alpha f(x,t) \tag{7}$$

can be viewed as the p-adic analogue of a scale-space equation, where t plays the role of the scaling parameter.

The Vladimirov operator is usually defined on L_2-functions by

$$D^\alpha \colon f \mapsto F M_{|x|_p^\alpha} F^{-1}(f),$$

where $\alpha \neq -1$, M_λ means multiplication with λ, and F is the Fourier transform. For $\alpha > 0$ it has the form (6) [21].

Let now $\Sigma \subseteq \mathbb{Q}_p^2$ be a p-adic image manifold. Via the linear isomorphism $\mathbb{Q}_p^2 \cong K$ for the unramified extension field $K = \mathbb{Q}_p(\theta)$ over $\mathbb{Q}_p$ of degree two, we will derive a one-dimensional description of p-adic scale space.

The real embedding of image space Σ can be given by

$$X \colon \Sigma \to \mathbb{R}^3, \quad \sigma = x + y\theta \mapsto (M(x), M(y), I(\sigma)),$$

where

$$M \colon \mathbb{Q}_p \setminus \left\{ \frac{a}{p^n} \mid a \in \mathbb{Z}_{\geq 1},\ n \in \mathbb{Z}_{\geq 0} \right\} \to \mathbb{R}, \quad \sum x_\nu \mapsto \sum x_\nu p^{-\nu}$$

is the *(inverse) Monna map*, provided the coordinates of all $\sigma \in \Sigma$ lie in the domain of M. We would like to develop an analogue of (3) in this situation, but contend ourselves with the Polyakov action on the Bruhat-Tits tree T_K from [22]:

$$S_{T_K}[X^i] = \frac{\beta_K}{2} \sum_{e \in E_K} \left(\partial_e X^i\right)^2$$

with $\partial_e X^i = X^i(t(e)) - X^i(o(e))$, and a p-adic "string tension" constant $\beta_K = 1/\ln q$, where $q = p^f$ is the cardinality of the residue field $\mathbb{F}_q$ of the unramified p-adic extension field K. By applying his method of integrating out the interior of T_K, one obtains

$$S_K[X^i] = \frac{q(q-1)\beta_K}{4(q+1)} \int\limits_\Sigma d\xi\, d\xi' \frac{\left(X^i(\xi) - X^i(\xi')\right)^2}{|\xi - \xi'|_K^2},$$

just as in the case $q = p$.

The variational derivative is

$$\frac{\delta S_K[X^i]}{\delta X^i}(\xi) = \mathcal{D}_K X^i(\xi) = \frac{q(q-1)\beta_K}{2(q+1)} \int\limits_\Sigma d\xi' \, \frac{X^i(\xi) - X^i(\xi')}{|\xi - \xi'|_K^2},$$

so that we propose the p-adic scale space equation

$$\partial_t X = \frac{\delta S_K[X]}{\delta X}$$

which yields in our case $\partial_t X = \mathcal{D}_K X$, the p-adic diffusion.

For complex-valued images, we propose

$$i\partial_t \Psi = \Delta_g \Psi, \quad \text{resp.} \quad i\partial_t \Psi = \mathcal{D}_K \Psi$$

as the scale space equations in the archimedean, resp. non-archimedean case. This allows to view phase-intensity images, as provided e.g. by range-image cameras or other active sensors, as wave-functions.

Notice that in the p-adic situation, the coordinates as well as the scaling parameter t are p-adic, which poses no problems in applications, as t then runs through a finite set. More on p-adic pseudo-differential equations, functional analyis and mathematical physics can be found in [21], an overview of applications to string theory is given by Brekke and Freund [7].

5 Hierarchical topological models

The purpose of image segmentation is to partition the image into homogeneous segments. From these segments one can construct a model of the observed scene, e.g. by labelling the individual segments as being part of some, possibly categorised, entity, such as 'building', 'person' or 'background'. Important for understanding images are their topological models which can be derived from segmentations. For example, the adjacency graph is often used in cartography. Here, the vertices are given by the segments, and an edge means that two segments are adjacent. Other topological types are also used, such as planar graphs which can model the borders between 'areas' as 'lines', and such 'lines' border in 'points'. In principle, any kind of finite topological space is conceivable as a topological model, possibly with further structure. Recently, *topological databases* have been introduced to the modelling of geographic or building information, together with *relational chain complexes* as the database notion of topological spaces and chain complexes, respectively [6].

A hierarchy of segmentations means that segments can be further segmented, resulting in a family of segmentations at various scales. Here, a segmentation at scale t is refined to one at the next scale $t+1$ in such a way that each parent segments (from scale t) is further partitioned into segments, and each segment at scale $t+1$ has 'kin segments' such that the union of all 'kins' is a 'parent' segment.

A corresponding topological model is then given by a tree of graphs, planar graphs or other topological spaces (or databases). The hierarchy from bottom to top induces continuous maps between those models at different scales. Through p-adic encoding of the bottom segmentation level (i.e. in some p-adic field K) one arrives at topologies on finite sets X of p-adic numbers. The hierarchical tree is thus embedded into the Bruhat-Tits tree T_K by taking the minimal subtree of T_K whose boundary contains X. Such embedded p-adic dendrograms were introduced to data analysis in [3], and are well suited for hierarchical data. In case $X \subseteq O_K$, where O_K is the ring of integers in K, the topological model of X at level n can be extracted from X modulo m_K^n, where m_K is the maximal ideal of O_K. If the bottom-up encoding is used, the methods from Section 4 can be applied to segmentation hierarchies.

Viewing the case of chain complexes more closely, let $C = C(X)$ be a (homological) chain complex over a commutative unitary ring R with a finite basis $X \subseteq {O_K}^2$, where $X = X_0 \sqcup \cdots \sqcup X_m$ is the partition into sets X_i of 'cells' x of 'dimension' $i = \dim(x)$. Let

$$\partial\colon C_i(X) = RX_i \to C_{i-1}(X) = RX_{i-1}$$

be the boundary operator, where

$$RY := \bigoplus_{y \in Y} Ry$$

is the free R-module over the set Y. The set at level ν is defined as the set of disks

$$X^{(\nu)} := \left\{ x^{(\nu)} := x + m_K^\nu \mid x \in X \right\}$$

with grading given by

$$\dim(x^{(\nu)}) := \max\left\{ i \mid y \in X_i,\ y \equiv x \mod m_K^\nu \right\},$$

a well defined notion. For each level ν there is the contraction map

$$\pi_\nu\colon X \mapsto X^{(\nu)}, \quad x \mapsto x^{(\nu)},$$

for which we require that any fibre $\pi_\nu^{-1}(x^{(\nu)})$ is open in the n-skeleton of $C(X)$, where $n = \dim(x^{(\nu)})$. By viewing the segment $x^{(\nu)}$ as a chain of its n-dimensional constituents in X (there are some choices for coefficients!), this induces from π_ν the linear map $\psi_\nu\colon C^\nu(X) = RX^{(\nu)} \to C(X)$ which maps n-chains to n-chains.

The requirement for the boundary map at level ν is

$$\partial^{(\nu)}\psi_\nu = \psi_\nu\partial,$$

in other words, the cellularity condition for ψ_ν. Notice that the directions of arrows ψ_ν and π_ν are reverse. As a result, we have obtained a hierarchical family of chain complexes 'modulo m_K^ν' from a given 'p-adic' chain complex. The same construction works for cohomological chain complexes as well.

However, the existence of $\partial^{(\nu)}$ is not guaranteed. If this is the case, we speak of *integrability* with respect to the linear map ψ_ν.

Definition 1. *The boundary operator ∂ is called* integrable *with respect to the linear map ψ_ν (or for short: ψ_ν-*integrable*), if there exists a boundary operator $\partial^{(\nu)}$ on $C(X^{(\nu)})$ such that ψ_ν is a cellular map of chain complexes. If the boundary operator ∂ is ψ_ν-integrable for all ν, then the collection $\psi = (\psi_\nu)$ is called a* hierarchical chain complex *for X, and ∂ is called ψ-*integrable*.*

[2] The top-down encoding is chosen for convenience only. Bottom-up works equally well.

The question is now, which conditions the linear maps in ψ must fulfill in order for ∂ to be ψ-integrable. Here, the map $\pi_\nu \colon X \to X^{(\nu)}$ plays an important role. In general, the fibre $\pi_\nu^{-1}(x)$ contains also cells of dimension lower than $n = \dim(x)$. These are, by assumption, contained in the interior of x. Assume that c is such an interior cell, and that $\dim(c) = n - 1$. Being in the interior of x means that c does not appear in the boundary $\partial\psi_\nu(x)$ in its representation as a linear combination of $n-1$-cells.

For a more systematic approach, we introduce some notation. The unknown boundary coefficients of x under $\partial^{(\nu)}$ will be written as $(x : c)_\nu$, where c runs through the cells of $X^{(\nu)}$:

$$\partial^{(\nu)}\, x = \sum_c (x : c)_\nu\, c.$$

The coefficients of a chain $d \in C_n(X)$ are denoted as $\langle d \mid y \rangle$, where y runs through the cells of X. Hence, we have the expression

$$\psi_\nu(x) = \sum_y \langle \psi_\nu(x) \mid y \rangle\, y = \sum_{y \in \pi_\nu^{-1}(x)} \langle \psi_\nu(x) \mid y \rangle\, y. \tag{8}$$

And for a cell y of X, the expression $(y : b)$ denotes the coefficient of the given boundary operator ∂, which yields:

$$\partial y = \sum_b (y : b)\, b.$$

From the chain map definition

$$\partial\psi_\nu(x) = \psi_\nu \partial^{(\nu)}(x) \tag{9}$$

we obtain for the left hand side:

$$\partial\psi_\nu(x) = \partial\left(\sum_{y \in \pi_\nu^{-1}(x)} \langle \psi_\nu(x) \mid y \rangle\, y\right) = \sum_{y \in \pi_\nu^{-1}(x)} \langle \psi_\nu(x) \mid y \rangle \sum_b (y : b)\, b,$$

where the linearity of ∂ was used. Changing the summation order in the last expression yields:

$$\partial\psi_\nu(x) = \sum_b \left(\sum_{y \in \pi_\nu^{-1}(x)} \langle \psi_\nu(x) \mid y \rangle (y : b)\right) b. \tag{10}$$

For the right hand side of (9), we obtain

$$\begin{aligned} \psi_\nu \partial^{(\nu)}(x) = \psi_\nu\left(\sum_c (x : c)_\nu\, c\right) &= \sum_c (x : c) \sum_{b \in \pi_\nu^{-1}(c)} \langle \psi_\nu(c) \mid b \rangle\, b \\ &= \sum_b (x : \pi_\nu(b)) \langle \psi_\nu(\pi_\nu(b)) \mid b \rangle\, b \end{aligned} \tag{11}$$

The result is:

Theorem 2. *The boundary operator ∂ is ψ_ν-integrable, if and only if for all $x, c \in X^{(\nu)}$ the equations*

$$(x : c)_\nu \langle \psi_\nu(c) \mid b \rangle = \sum_{y \in \pi_\nu^{-1}(x)} \langle \psi_\nu(x) \mid y \rangle (y : b) \tag{12}$$

have a common solution $(x : c)_\nu$ for all b having the same image $\pi_\nu(b) = c$.

Notice that, due to (8), the right hand side of (12) can be also written as a summation accross all $y \in X$.

The following corollary is an algebraic way of saying that any $n-1$-dimensional part of an n-cell x lies in the interior of x.

Corollary 3. *Assume that there is some $e \in \pi_\nu^{-1}(x)$ with $\dim(e) = n - 1$ and $x \in X_\nu$ with $\dim(x) = n$. If ∂ is ψ_ν-integrable, then $\langle \partial\psi_\nu(x) \mid e \rangle = 0$.*

Proof. The statement is valid, because

$$\begin{aligned} \langle \partial\psi_\nu(x) \mid e \rangle &= \sum_b \left(\sum_{y \in \pi_\nu^{-1}} \langle \psi_\nu(x) \mid y \rangle (y : b) \right) \langle b \mid e \rangle && (13) \\ &= \sum_{y \in \pi_\nu^{-1}(x)} \langle \psi_\nu(x) \mid y \rangle (y : e) && (14) \\ &= (x : x) \langle \psi_\nu(x) \mid e \rangle && (15) \\ &= 0 && (16) \end{aligned}$$

The first equality (13) follows from (10). The next, (14), follows from the fact that

$$\langle b \mid e \rangle = \begin{cases} 1, & \text{if } b = e \\ 0, & \text{otherwise.} \end{cases}$$

Equality (15) follows from Theorem 2. The last equality (16) follows, because of $(x : x) = 0$. □

So far, we have not used the assumption that the tree underlying ψ is p-adic, i.e. embedded into the Bruhat-Tits tree T_K for some p-adic field K. The vertices all correspond to some p-adic disks in K, and the labels coming from ψ can be viewed as locally constant integer valued functions. In other words, the $X^{(\nu)}$ each are sets of disjoint disks, and the boundary operators $\partial^{(\nu)}$ define functions on $K \times K$:

$$D^\nu = \sum_{(b,c) \in X^{(\nu)} \times X^{(\nu)}} \frac{(b : c)_\nu}{\mu(c) \cdot \mu(b)} 1_{b \times c},$$

where 1_A denots the indicator of a set A, and $\mu(d)$ the diameter of disk d. We call D^ν the *p-adic boundary function associated to $\partial^{(\nu)}$*. This yields

$$(b : c)_\nu = \int_b \int_c D^\nu(x, y)\, dx dy,$$

as can be readily verified. The maps ψ_ν define functions on K:

$$P^\nu(x,y) = \sum_{(b,c)\in X^{(\nu)}\times X^{(\nu)}} \frac{\langle \psi_\nu(b) \mid c\rangle}{\mu(b)\mu(c)} 1_{b\times c},$$

and these yield

$$\langle \psi_\nu(b) \mid c\rangle = \int\limits_b \int\limits_c P^\nu(x,y)\, dxdy,$$

just like above. The p-adic formulation of Theorem 2 can now be stated:

Theorem 4. *∂ is ψ_ν-integrable if and only if for all $c \in X^{(\nu)}$*

$$\int\limits_c D^\nu(x,y)\, dy \int\limits_c P^\nu(x,y)\, dy = \int\limits_K P^\nu(x,y) D(x,y)\, dy,$$

where $D = D^\mu$ is the p-adic boundary function corresponding to the boundary operator at the highest level of resolution.

Proof. Assume that $x \in A$, $z \in b$ with $A, b \in X^{(\nu)}$. Then the left hand side equals

$$\frac{(A:c)_\nu \,\langle \psi_\nu(c) \mid b\rangle}{\mu(A)\,\mu(b)},$$

and the right hand side:

$$\sum_{B\in X_\nu} \frac{\langle \psi_\nu(A) \mid B\rangle\,(B:b)}{\mu(A)\,\mu(b)},$$

and the assertion is a consequence of Theorem 2. □

Corollary 5. *Assume that there exists $e \in \pi_\nu^{-1}(A)$ for some $A \in X^{(\nu)}$ with* $\dim e = \dim A - 1$. *If ∂ is ψ_ν-integrable, then for all $x \in A$ and $z \in e$ it holds true that*

$$\int\limits_K P^\nu(x,y) D(y,z)\, dy = 0.$$

Proof. From Theorem 4 it follows that

$$\int\limits_K P^\nu(x,y) D(x,y), dy = \int\limits_A D^\nu(x,y)\, dy \int_A P^\nu(y,z)\, dy = (A:A)_\nu \,\langle \psi_\nu(A) \mid e\rangle = 0,$$

which proves the assertion. □

In analogy to before, Corollary 5 is a p-adic way of saying that e is an 'inner' cell of A.

Notice that, in the integrable case, ψ is a new instance of a *persistence chain complex* associated to the hierarchical data X, allowing for the computation of a topological barcode for X which represents the 'lifetimes' of homology generators along the level ν of the hierarchy [8]. In our case, with any kind of p-adic encoding of X, we call ψ, as well as the family (P^ν, D^ν), a *p-adic persistence chain complex* or a *persistence chain complex over K*, if we want to emphasise the p-adic field of definition.

6 Concluding remarks

Viewing the hierarchical world as ultrametric leads to the consideration of p-adic methods for detecting and processing hierarchies. For this, p-adic data encoding becomes indispensible. This applied to images yields encodings of special quadtrees, known in image processing. The bottom-up method introduced here opens the way for methods from p-adic mathematical physics, whereas the top-down method renders p-adic integers as image coordinates. The former enabled to derive a p-adic scale space equation from the Polyakov action on the Bruhat-Tits tree. And the latter recently allowed the use of Hensel's lemma to the equations arising in the problem of finding the essential matrix from five point-correspondences in stereo vision [5].

Applied to deriving topological models from segmentation, we showed how to derive a hierarchy of topologies (which cartographers call *generalisation* or *multi-representation*) from a given topology on the p-adic code. This has been seen to apply to chain complexes as well, where the existence of the boundary operators on all levels depends on an *integrability condition* coming from the chain map rule. This condition has a formulation with p-adic integrals. Hence, the boundary operators can be computed from the highest level of resolution by means of p-adic integration whenever the corresponding inter-level chain map is injective.

We note that an axiomatic p-adic scale-space theory, from which feature detectors and descriptors can be derived, has yet to be developped. In any case, the results show that p-adic physics can play an important role in the understanding of images.

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