CYCLIC COVERINGS OF THE p-ADIC PROJECTIVE LINE BY MUMFORD CURVES

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Abstract. Exact bounds for the positions of the branch points for cyclic coverings of the p-adic projective line by Mumford curves are calculated in two ways. Firstly, by using Fumiharu Kato's $*$-trees, and secondly by giving explicit matrix representations of the Schottky groups corresponding to the Mumford curves above the projective line through combinatorial group theory.

1. Introduction

Cyclic covers of the projective line defined over a field K of characteristic zero have been thoroughly studied. Such covers $\varphi\colon X \to \mathbb{P}^1$ correspond to equations of the form

$$y^n = f(x),$$

where $f(x) \in K[x]$ is a polynomial. The zeros of $f(x)$ in a suitable finite extension field of K are the branch points of φ. In the case that K is a p-adic field, it is known that not every equation as above corresponds to a cover by a Mumford curve. And even if $f(x)$ is of the right kind, one finds strong restrictions on the position of the branch points for φ to be a Mumford cover of $\mathbb{P}^1$. This was first observed in the case $n = 2$ and X an elliptic curve: X is a Tate curve, if and only if the four branch points do not form an equilateral quadrangle in $\mathbb{P}^1$. To be more precise, by a projective automorphism one can take the branch locus to be $\{0, 1, \infty, \lambda\}$ with $|\lambda| = 1$. Then for residue characteristic not equal to two, the Legendre equation

$$y^2 = x(x-1)(x-\lambda)$$

is the equation of a Tate curve, if and only if $|\lambda - 1| < 1$.

If φ is a hyperelliptic cover, then the restriction found in [16] is that the branch points come in pairs of points closer to one another than to the other branch points. The distance is measured by rational affinoid subsets of $\mathbb{P}^1$.

The most elegant way of obtaining bounds for relative positions of the branch points of any finite Galois cover φ of $\mathbb{P}^1$ is through the $*$-tree $\mathscr{T}_N^*$ for the discrete finitely generated group N giving rise to the orbifold uniformisation of $\Omega \xrightarrow{N} \mathbb{P}^1$ factoring through φ and having the same branch locus and ramification orders as φ. Distances within $\mathscr{T}_N^*$ translate into distances between branch points. The group N sits in an exact sequence

$$1 \to \Gamma \to N \to G \to 1,$$

where G is the Galois group of φ and Γ is a free group whose rank is the genus of X. The corresponding uniformisation $\Omega \xrightarrow{\Gamma} X$ is called a *Schottky uniformisation*,

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and Γ a *Schottky group*. This approach is pursued in the present article for cyclic covers.

The $*$-tree $\mathscr{T}_N^*$ was developped by Fumiharu Kato in order to obtain deeper insight into the structure of p-adic discrete groups and has many applications in the study of automorphisms of Mumford curves, especially in positive characteristic, e.g. [5]. Extensive use of the $*$-tree is being made in the classification of p-adic triangle groups [3].

In the present article, we focus on all possible cyclic covers of $\mathbb{P}^1$ with twofold aim.

Firstly, we exhibit detailed calculations of the exact bound for $|\lambda-1|$ characterising the Mumford covers among four-point cyclic Harbater-Mumford covers φ whose branch locus is $\{0,1,\infty,\lambda\}$ and $|\lambda|=1$, and give the exact sizes of the separating annuli for covers with more branch points.

Secondly, explicit hyperbolic generators for Γ are given from which again one can calculate the characteristic bound from above by Ford's method of isometric circles and some combinatorial group theory, and thus gains an explicit parametric description of the Schottky uniformisation of the Mumford curve. For covers of prime degree, we recover the generators of [17].

We remark, however, that here we are only dealing with the positions of branch points up to "first order", meaning that our methods do not reveal the precise relationship between the discrete representation $\Gamma \to \mathrm{PGL}_2(K)$ and the branch points of the corresponding Mumford cover φ, which would require the study of automorphic forms on the Mumford curves. Geometrically speaking, we can make explicit the geometry of $\mathscr{T}_N^*$ without, however, considering the precise embeddings of $\mathscr{T}_N^*$ into the Bruhat-Tits tree for $\mathrm{PGL}_2(K)$.

A nice desideratum would be the explicit fuchsian differential equation corresponding to the cyclic cover $\varphi\colon X \to \mathbb{P}^1$.

This article refines methods and results from the author's dissertation [1].

2. Generalities

Let $\mathbb{Q}_p$ be the field of p-adic numbers. We assume that K is a finite extension field of $\mathbb{Q}_p$, large enough that all branch points of all covers $X \to \mathbb{P}^1_K$ in the article are K-rational.

$\mathscr{T}_K$ denotes the Bruhat-Tits tree for $\mathrm{PGL}_2(K)$, the automorphism group of the projective line $\mathbb{P}^1_K$. We will use the well known fact that the ends of $\mathscr{T}_K$ correspond to the K-rational points of the projective line $\mathbb{P}^1$.

Let $N \subseteq \mathrm{PGL}_2(K)$ be a finitely generated discrete subgroup. Following [13], the tree $\mathscr{T}_N^*$ is defined to be the smallest subtree of $\mathscr{T}_K$ whose ends correspond to the fixed points of all non-trivial elements of N. The group N acts on $\mathscr{T}_N^*$ without inversion, and the quotient graph $T_N^* = \mathscr{T}_N^*/N$ is a graph of finite groups with finitely many ends corresponding to the branch points of the quotient cover

$$\Omega_N \xrightarrow{N} X_N.$$

The open analytic space $\Omega_N \subseteq \mathbb{P}^1$ is defined as the complement of the closure of the limit points of N, and the quotient space X_N is the analytification of a non-singular projective algebraic curve over K. In fact, X_N is a Mumford curve.

An important example of $\mathscr{T}_N^*$ is, when $N = \langle\gamma\rangle \cong C_m$ is a finite cyclic group of order $m > 1$. Then $M(\gamma) := \mathscr{T}_N^*$ is simply a straight line stabilised by γ.

Definition 2.1. *Let $\gamma \in \mathrm{PGL}_2(K)$ be of finite positive order. Then $M(\gamma)$ is called the* mirror *of γ.*

Lemma 2.2. *There is a natural bijection between the sets:*

$$\{\textit{maximal finite cyclic subgroups of } N\} \xrightarrow{\sim} \{\textit{mirrors of } N\}$$

Proof. The natural map takes a maximal cyclic group to the mirror of a generator, which is clearly well defined. It is also clear that $\langle\gamma\rangle \subseteq \langle\delta\rangle$ implies $M(\gamma) = M(\delta)$. Therefore, the map is surjective.

Let now $\langle\gamma\rangle, \langle\delta\rangle \subseteq N$ be such that the mirrors $M := M(\gamma) = M(\delta)$ coincide. Then $G := \langle\gamma, \delta\rangle \subseteq N$ is finite, as any word in γ and δ fixes the mirror. This means that $M = \mathscr{T}_G^*$, implying that G is cyclic, as the corresponding cover $\mathbb{P}^1 \xrightarrow{G} \mathbb{P}^1$ has exactly two branch points [13, Proposition 5.6.2]. □

It is well known that, if K'/K is a finite field extension, then a subdivision of $\mathscr{T}_K$ embeds into $\mathscr{T}_{K'}$.

Convention 2.3. *Let $\mathscr{T}$ be a subtree of $\mathscr{T}_K$. When we speak of a* point *x on an edge $e = (v, w)$ of $\mathscr{T}$, we mean that after some finite extension K'/K, x is a vertex on the (open)* path *(v, w) in the tree $\mathscr{T}'$ obtained by restricting the subdivision and embedding process from above.*

3. Mumford curves and discrete groups

3.1. **The tree $\mathscr{T}_\Gamma^*$ for a free product of cyclic groups.** Let $\Gamma = C_m * C_n$ be the free product of two cyclic groups C_m and C_n. We will calculate the tree $\mathscr{T}_\Gamma^*$ for all possible values of $m = p^r a$ and $n = p^s b$ (where $(a, p) = (b, p) = 1$). In fact, the shape of the quotient tree T_Γ^* is known in [13, §8.1] (and implicitly known in [9, §11]) and is given in Figure 1.

Our special interest lies in the exact distances within the tree, in particular the lengths of the paths $[x, v]$ and $[w, y]$. These can be extracted from the indications at the end of [13, §8.1] or the proof of [2, Proposition 3.1]. Here, we give a detailed exposition of the calculations.

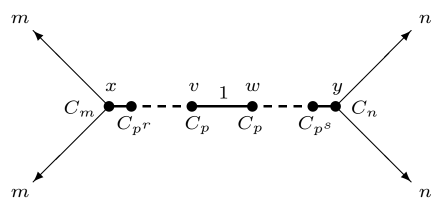


Figure 1. The quotient $*$-tree for a free product of cyclic groups.

Proposition 3.1. *Let $\Gamma \cong C_m * C_n$ be a discrete subgroup of $\mathrm{PGL}_2(K)$. If $(m, p) = (n, p) = 1$, then T_Γ^* is as in Figure 1 with $\mathrm{dist}(x, v) = \mathrm{dist}(w, y) = 0$.*

Proof. [13, §8.1]. □

Proposition 3.2. *Let* $\Gamma_1 \subseteq \Gamma_2$ *be discrete subgroups of* $\mathrm{PGL}_2(K)$, *where as abstract groups* $\Gamma_1 \cong C_p * C_p$ *and* $\Gamma_2 \cong C_{pa} * C_{pb}$ *with* $a, b \geq 1$. *Then:*

(1) *There is a subdivision* $\mathscr{T}_1^*$ *of* $\mathscr{T}_{\Gamma_1}^*$ *which is a subtree of* $\mathscr{T}_{\Gamma_2}^*$.
(2) *The quotient graphs* $T_{\Gamma_1}^*$ *and* $T_{\Gamma_2}^*$ *are trees with shape of that in Figure 1.*
(3) *For a primitive p-th root* ζ_p *of unity,*

$$\mathrm{dist}(x, v) = \mathrm{dist}(w, y) = v(\zeta_p - 1)$$

in both trees $T_{\Gamma_1}^*$ *and* $T_{\Gamma_2}^*$.

Proof. Since Γ_2 is a free product of non-trivial cyclic groups, $\mathscr{T}_{\Gamma_2}^*$ contains an edge $e = (v, w)$ with stabilisers $\Gamma_{2,e} = 1$ and $\Gamma_{2,v}$, $\Gamma_{2,w}$ both non-trivial. The group $\Gamma_{2,v} = \langle\gamma\rangle$ contains an element of order p, as otherwise v would lie on the mirror $M(\gamma)$ which in turn corresponds to some maximal finite cyclic subgroup of Γ_2 (Lemma 2.2). But such subgroups necessarily contain elements of order p, a contradiction.

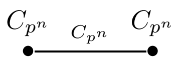
Let $1 \neq \gamma \in \Gamma_{2,v}$. Then v does not lie on the mirror $M(\gamma)$, as otherwise any point on e close enough to v would be stabilised by γ. As this holds for all $\gamma \neq 1$ in $\Gamma_{2,v}$, we conclude that $\Gamma_{2,v} \cong C_p$. Therefore, the vertex v has to $M(\gamma)$ the positive distance $v(\zeta_p - 1)$ [11, Lemma 3]. Analogously, $\Gamma_{2,w}\langle\delta\rangle \cong C_p$ and the distance between w and $M(\delta)$ is also $v(\zeta_p - 1)$.

Taking $e \subseteq \mathscr{T}_{\Gamma_1}^*$, we have $M(\gamma) \cup M(\delta) \subseteq \mathscr{T}_{\Gamma_1}^*$. From this, all three assertions follow. □

Proposition 3.3. *Let* $\Gamma \cong C_{pm} * C_n$ *with* $(n, p) = 1$ *be a discrete subgroup of* $\mathrm{PGL}_2(K)$. *Then* T_Γ^* *is as in Figure 1 with* $\mathrm{dist}(w, y) = 0$.

Proof. The proof is similar to that of Proposition 3.2. □

Remark 3.4. *The figure in* [2, Fig. 1] *is slightly erroneous. It should have two segments stabilised by* C_{p^n}



which are not contained in the two mirrors. This can be seen by setting $a = b = 1$, *and* $r = s = n$ *in Figure 1.*

3.1.1. *Some examples.* Figures 2, 3 and 4 show portions of some $\mathscr{T}_\Gamma^*$ for $p = 2$, where $\mathrm{dist}(x, v) \neq 0$ (notation as in Figure 1). We remark, however, that the most beautiful $*$-trees are those for the finite groups, when $p = 2, 3, 5$ (to appear in [3]), some of which are illustrated already in [4].

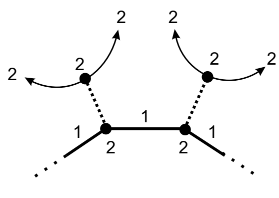


FIGURE 2. The tree $\mathscr{T}_\Gamma^*$ for $\Gamma = C_2 * C_2$ and $p = 2$.

The figures are depicted in such a way that

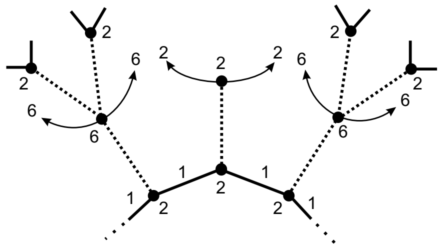


FIGURE 3. The tree $\mathscr{T}_\Gamma^*$ for $\Gamma = C_2 * C_6$ and $p = 2$.

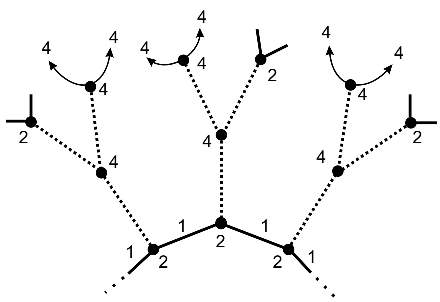


FIGURE 4. The tree $\mathscr{T}_\Gamma^*$ for $\Gamma = C_4 * C_4$ and $p = 2$.

- a curved line with arrow heads represents the mirror of a transformation whose order is written at both ends;
- an unbroken line segment denotes an edge lying on a geodesic line on which a hyperbolic element acts through translation;
- a dotted line segment means a non-trivially stabilised edge which does not lie on a mirror (the order of whose stabiliser is the lower of the two numbers at its extremities);
- a number is the order of the stabiliser of the corresponding vertex or edge.

3.1.2. *The positions of ends.* Let $a = (a_0 : a_1)$, $b = (b_0 : b_1)$, $c = (c_0 : c_1)$ and $d = (d_0 : d_1)$ be four pairwise distinct K-rational points of $\mathbb{P}^1_K$. The arrangement of two straight lines (a, b) and (c, d) in $\mathscr{T}_K$ can be calculated using the crossratio

$$R(a, b; c, d) = \frac{(a_1c_0 - a_0c_1)(b_1d_0 - b_0d_1)}{(a_0b_1 - a_1b_0)(c_0d_1 - c_1d_0)}.$$

Proposition 3.5. *Let $a, b, c, d \in \mathbb{P}^1_K$ be as above. Then:*

(1) *If $|v(R(a, b; c, d))| = |v(R(b, a; c, d))| = 0$, then (a, b) and (c, d) intersect at exactly one vertex.*
(2) *If $|v(R(a, b; c, d))| = |v(R(b, a; c, d))| \neq 0$, then (a, b) and (c, d) are disjoint with the distance $|v(R(a, b; c, d))|$.*
(3) *If $|v(R(a, b; c, d))| \neq |v(R(b, a; c, d))|$, then the intersection of (a, b) and (c, d) is the path $[v(a, b, c), v(b, c, d)]$ of length*

$$\max\{|v(R(a, b; c, d))|, |v(R(b, a; c, d))|\}.$$

Here, $v(a, b, c)$ denotes the unique vertex in $\mathscr{T}_K$ determined by the points $a, b, c \in \mathbb{P}^1_K$ viewed as ends in $\mathscr{T}_K$.

Proof. [13, Proposition 3.5.1]. □

Definition 3.6. *Let ζ be a primitive m-th root of unity. Then ε_m and $\alpha_p(m,n)$ denote the numbers*

$$\varepsilon_m := \begin{cases} 1, & \text{if } p \mid m \\ 0, & \text{otherwise} \end{cases}$$

$$\alpha_p(m,n) := |1-\zeta_p|^{\varepsilon_m+\varepsilon_n}.$$

T_Γ^* has four ends which can be taken as $0,\infty$ going out of x and $1,\lambda$ emanating from y with $|\lambda|=1$.

Theorem 3.7. *Let $\Gamma \cong C_m * C_n$ be a discrete subgroup of $\mathrm{PGL}_2(K)$ and $0,\infty;1,\lambda$ the ends of T_Γ^* as above. Then:*

$$|\lambda - 1| < \alpha_p(m,n),$$

and, conversely, for all such $\lambda \in K$ there is an embedding $C_m * C_n \to \mathrm{PGL}_2(K)$ as a discrete subgroup having such a $*$-tree.

Proof. Let Γ be as stated. Then we have, by Propositions 3.1, 3.2 and 3.3,

$$\begin{aligned}\mathrm{dist}(x,y) &= \mathrm{dist}(x,v) + \mathrm{dist}(v,w) + \mathrm{dist}(w,y) \\ &> \begin{cases} 2\cdot v(\zeta_p - 1), & \text{if } p \mid m \text{ and } p \mid n, \\ v(\zeta_p - 1), & \text{if } p \mid m \text{ and } (p,n)=1, \\ 0, & \text{otherwise,}\end{cases}\end{aligned}$$

as $d(v,w)$ is strictly positive. Thus, by Proposition 3.5, it holds true that

$$|v(\lambda-1)| = \mathrm{dist}(x,y),$$

as $|v(R(0,\infty;1,\lambda))| = |v(R(\infty,0;1,\lambda))| = |v(\lambda-1)| \neq 0$. From this, the assertion follows.

For the converse implication, one has to check that the $*$-tree from Figure 1 is realisable for any value of $\mathrm{dist}(v,w) > 0$ in $|v(K^\times)|$. This easy task is left to the reader. □

Example 3.8. For $\Gamma \cong C_2 * C_2$, we obtain the realisability for $\mathscr{T}_\Gamma^*$ if and only if $|\lambda - 1| < |2|^2$. In this case, the projective line is covered by a Tate elliptic curve. By the formula for the j-invariant of elliptic curves [8, Chapter IV.4],

$$j = 2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2},$$

this is equivalent to $|j| > |2|^4$.

3.2. **Cyclic Mumford covers.** The only cyclic covers $X \xrightarrow{m} \mathbb{P}^1$ allowing X to be a Mumford curve are known to be those corresponding to an equation of the form

$$y^m = \prod_{i=1}^{r} (x-\lambda_{i1})^{a_i}(x-\lambda_{i2})^{m-a_i}. \tag{1}$$

The branch points of the cover are the zeros of the polynomial on the right hand side. After some projective K-linear transformation, the first four terms can be taken as $x^{a_1}, 1, (x-1)^{a_2}, (x-\lambda)^{m-a_2}$, corresponding to the branch points $0,\infty;1,\lambda$. We will call a cover whose equation is of the form (1), a *cover of HM-type*.

Definition 3.9. *By an* m-cover *of type* $(e_1, \dots, e_r)$ *we mean a cyclic cover* $\varphi\colon X \to \mathbb{P}^1$ *of degree* m *of HM-type ramified above the points* $(\lambda_{11}, \lambda_{12}; \dots; \lambda_{r1}, \lambda_{r2})$ *with* $(\lambda_{11}, \lambda_{12}; \lambda_{21}, \lambda_{22}) = (0, \infty; 1, \lambda)$ *and* $|\lambda| = 1$*, and such that the ramification index above each* λ_{ij} *is* $e_i > 0$*.*

A cyclic cover $X \to \mathbb{P}^1$ *is called a* Mumford cover *or* of Mumford type*, if* X *is a Mumford curve.*

Definition 3.10. *The statement*

The bound holds for m.

means saying that an m*-cover of type* (d, e) *is a Mumford cover if and only if* $|\lambda - 1| < \alpha_p(m, n)$*.*

Theorem 3.11. *The bound holds for* m*.*

Proof. This, of course, is an immediate consequence of Theorem 3.7. However, the statement will be proven again in Section 5 by different methods. ☐

3.3. **Free products of cyclic groups.**

Lemma 3.12. *Let* G *be a free product of finitely many groups* $G_1, \dots, G_r$*. Then each non-trivial element* s *of finite order lies in exactly one conjugate of one of the factors* G_i*.*

Proof. By [14, IV.1.6], s lies only in conjugates of some of the G_i. Assume therefore that $s \in G_1$. Then the equation $s = g^{-1} s_i g$ with $s_i \in G_i$ and $i \neq 1$ is easily seen to lead to a contradiction. ☐

Let $N = \langle s_0 \rangle * \dots * \langle s_m \rangle \subseteq \mathrm{PGL}_2(K)$ be the m-fold free product of the cyclic group C_n acting discontinuously on $\mathbb{P}^1$. By the universal property of free products, there is a unique homomorphism $\varphi\colon N \to C_n$ such that for each $i = 0, \dots, m$ the diagram

$$\begin{array}{ccc} & \langle s_i \rangle & \\ \swarrow & & \searrow \cong \\ N & \xrightarrow{\ \varphi\ } & C_n \end{array}$$

is commutative. This homomorphism φ depends on the choices of the isomorphisms $\langle s_i \rangle \to C_n$. Here, all s_i are supposed to be mapped to the same generator of C_n. In Section 5, we will consider also other choices.

Let $\Gamma := \ker \varphi \subseteq N$. It is easily seen to be of finite index n in N.

Proposition 3.13. *The group* Γ *is free of rank* $m(n-1)$ *and freely generated by*

$$s_0^j s_i s_0^{-j-1}, \qquad j = 1, \dots, n-1, \quad i = 1, \dots, m.$$

Proof. By [15, §1.3 and §2.(4)], Γ is generated by the asserted elements. This generating system cannot be shortened, as the genus g of the Mumford curve $X = \Omega_N/\Gamma$ can be calculated by the Riemman-Hurwitz formula for the cyclic cover

$$X \to \mathbb{P}^1$$

of degree n totally ramified in the Γ-orbits of the points in Ω_N fixed by some s_i. For this, we must check that the s_i have regular fixed points: this follows from [10, Satz 6], as, by Lemma 3.12, s_i fixes precisely one vertex of $\mathscr{T}_N$. So, we have $2m+2$ ramification points, and

$$2g-2=-2n+(2m+2)(n-1),$$

which is equivalent to

$$g=m(n-1).$$

Any non-trivial element $\gamma\in\Gamma$ of finite order is conjugated to an element of some $\langle s_i\rangle$. It follows that, in any representation of γ as a word in the generators $s_0^j s_i s_0^{-j-1}$ and their inverses, the sum of the exponents cannot be zero—a contradiction. Therefore, Γ is torsion-free. As Γ is the fundamental group of a tree of groups, it follows that Γ is free. □

4. Many-point Mumford covers

Lemma 4.1. *Let $N\subseteq \mathrm{PGL}_2(K)$ be a free tree product of finite cyclic groups $C_1,\dots,C_r$. Then N is discrete if and only if each free amalgam $C_i*C_j\subseteq N$ of two neighbouring factors of N is discrete.*

Proof. If N is discrete, then so is the subgroup C_i*C_j.

We prove the converse by induction on r. If $r=2$, then the statement clearly holds true. Let, for $r>2$, $N=N'*C$ with N' a free tree product with $r-1$ factors and $C=C_i$ for some i between 1 and r. By the induction hypothesis, N' is discrete. Also $C':=C*C_j$ is discrete, where C_j is the unique factor of N' neighbouring to C. Clearly, $T:=T^*_{N'}\cup T^*_{C'}\subseteq\mathscr{T}_K$ is a tree, and

$$\mathscr{T}:=\bigcup_{\gamma\in N}\gamma T\subseteq\mathscr{T}_K$$

is a tree upon which N acts with finite vertex stabilisers:

$$N_v\cong\begin{cases}N'_v, & v\in\gamma T^*_{N'}\\ C'_v, & v\in\gamma T^*_{C'}\end{cases}\quad\text{for some }\gamma\in N.$$

By [13, Lemma 4.4.1(2)], it follows that N is discrete. □

Remark 4.2. In fact, with the notations from the proof of Lemma 4.1, it holds true that

$$\mathscr{T}=\mathscr{T}^*_N.$$

This is due to the fact that N is generated by the stabilisers N_v, where v runs through all vertices of T (cf. the "if" part in "$g=0$" of the proof of Theorem II in [13, §7.]).

Lemma 4.1 allows us to prove a geometric criterion for arbitrary m-covers to be Mumford covers. For this, denote by $\mathrm{br}(\varphi)$ the branch locus of an m-cover.

Theorem 4.3. *An m-cover of type $(e_1,\dots,e_r)$ is a Mumford cover, if and only if, after a suitable re-ordering of the pairs (λ_{ij},e_i), there is an affinoid covering $\mathfrak{U}=\{U_1,\dots,U_r\}$ of $\mathbb{P}^1$ such that*

1. *$U_i\cap U_j$ is either empty or an annulus of thickness $\alpha_p(e_i,e_j)$, if $i\neq j$,*
2. *for all $i=1,\dots,r$ holds true: $U_i\cap\mathrm{br}(\varphi)=\{\lambda_{i1},\lambda_{i2}\}$.*

Proof. If an m-cover $\varphi\colon X \to \mathbb{P}^1$ is a Mumford cover, then $\mathbb{P}^1$ can be uniformised by a free tree product N of cyclic groups, i.e. φ is part of a commutative diagram

$$\begin{array}{ccc} \Omega & \longrightarrow & X \\ & {}_{N}\searrow & \downarrow{\scriptstyle \varphi} \\ & & \mathbb{P}^1 \end{array}$$

where X is a Mumford curve, and the space $\Omega \subseteq \mathbb{P}^1$ is the complement of the closure of the set of limit points for the action of N given by some discrete faithful representation $\tau\colon N \to \mathrm{PGL}_2(K)$.

The tree $\mathscr{T}_N^*$, embedded in $\mathscr{T}_K$ via τ, allows the extraction of the cover $\mathfrak{U}$: the stars around the vertices whose stabilisers are maximal yield discs U_i with $\deg(v)-1$ "holes", and the paths between any two nearest such vertices correspond to annuli whose thickness was calculated in the proof of Theorem 3.7 as $\alpha_p(e_i, e_j)$.

Let, conversely, φ be an m-cover satisfying the conditions (1) and (2). Taking two intersecting $U_i, U_j \in \mathfrak{U}$, we can construct a four-point m-cover by setting the ramification indices of all branchpoints outside $U_i \cup U_j$ to one. This is a Mumford cover, by Theorem 3.7. Doing this for all intersecting pairs of affinoids from $\mathfrak{U}$, we obtain a free amalgamated product which is discrete, by Lemma 4.1. □

Corollary 4.4. *A cyclic cover $\varphi\colon X \to \mathbb{P}^1$ is of Mumford type, if and only if there is some $\alpha \in \mathrm{PGL}_2(K)$ such that $\alpha \circ \varphi$ is an m-cover satifying conditions (1) and (2) of Theorem 4.3.*

Proof. This follows immediately from the fact that the cross-ratio of any four points in $\mathbb{P}^1$ is invariant under projective linear transformations. □

Remark 4.5. Theorem 4.3 generalises the characterisation from [16] of hyperelliptic Mumford curves among 2-covers, proven in the case of residue characteristic unequal 2 and by entirely different methods. This geometric condition is used by Frank Herrlich for constructing a moduli space of hyperelliptic Mumford curves [12].

5. The bound for four-point covers again

In the following, we will calculate the bound for four-point cyclic covers by giving explicit faithful representations $\tau\colon N = C_m * C_n \longrightarrow \langle s, t\rangle \subseteq \mathrm{PGL}_2(K)$. In the discrete case, the fixed points of s and t correspond then to four "upstairs" ramification points of the cover $\Omega_{\tau(N)} \xrightarrow{\tau(N)} \mathbb{P}^1$ which we may and will assume to be $0, \infty; 1, \lambda$ with $|\lambda| = 1$.

The following Lemma shows that this approach is indeed legitimate, albeit indirect.

Lemma 5.1. *Assume that the branch locus of φ is $0, \infty, 1, \lambda'$ with $|\lambda'| = 1$. Then it holds true that*

$$|\lambda - 1| < \alpha_p(m, n) \quad \Leftrightarrow \quad |\lambda' - 1| < \alpha_p(m, n).$$

Proof. This follows from the fact that any section $T_N^* \to \mathscr{T}_N^*$ is isometric. □

Remark 5.2. *In fact the first approach from Section 3 was indirect in the same manner as the approach in this section, as we calculated* T_N^* within $\mathscr{T}_N^*$.

The Kummer equations $y^m = x^a(x-1)^b(x-\lambda)^c$ *to follow are to be understood modulo Lemma 5.1. of the corresponding covers. In fact, the precise correspondence between discrete faithful representations of* N *and Kummer equations is still not settled.*

5.1. **Galois covers of prime degree.** Let $X \to \mathbb{P}^1$ be a cyclic cover of prime degree q totally ramified above exactly four points. By projective linear transformation, we may assume that the branch locus of the cover consists of the points 0, 1, ∞ and λ, where $|\lambda| = 1$. The aim of this section is to redetermine explicitly the conditions on λ for which X can be a Mumford curve by using Ford's isometric circles[1].

Let $N = N_{q,q} = \langle s\rangle * \langle t\rangle \subseteq \mathrm{PGL}_2(K)$ be the free product of two copies of the cyclic group C_q, where s is given by the matrix

$$s = \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix},$$

where ζ is a primitive q-th root of unity, and t is obtained from s by conjugation with

$$\varphi = \begin{pmatrix} \lambda & 1 \\ 1 & 1 \end{pmatrix}.$$

The latter means that

$$t = \varphi s \varphi^{-1} = \frac{1}{\lambda - 1}\begin{pmatrix} \lambda\zeta - 1 & \lambda(1-\zeta) \\ \zeta - 1 & \lambda - \zeta \end{pmatrix}.$$

The elliptic transformation s has the fixed points 0 and ∞, whereas t has the fixed points 1 and λ.

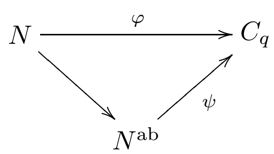
For further reference, we also give the matrix t^{-1}:

$$t^{-1} = \varphi s^{-1} \varphi^{-1} = \frac{1}{\lambda - 1}\begin{pmatrix} \lambda\zeta^{-1} - 1 & \lambda(1-\zeta^{-1}) \\ \zeta^{-1} - 1 & \lambda - \zeta^{-1} \end{pmatrix}.$$

Lemma 5.3. *The normal free subgroups* Γ *of* N *of index* q *are all of rank* $q-1$ *and given as* $\Gamma = \Gamma_f = \ker \varphi_f$ $(f = 1, \dots, q-1)$, *where each* φ_f *is the map*

$$\varphi_f \colon N \to C_q, \quad s \mapsto \zeta,\ t \mapsto \zeta^f.$$

Proof. The $\Gamma_f = \ker\varphi_f$ are clearly normal and, by Proposition 3.13, these groups are free of rank $q-1$. These are in fact all normal subgroups of index q, as every group homomorphism $\varphi\colon N \to C_q$ factorises through the abelian group $N^{\mathrm{ab}} = \langle s\rangle \times \langle t\rangle$:



and the map ψ can be made into the form

$$\psi_f \colon N^{\mathrm{ab}} \to C_q, \quad s \mapsto \zeta,\ t \mapsto \zeta^f$$

via an automorphism of C_q. □

[1]For the notion of *isometric circles* and their properties, cf. [6, Ch. I, §11].

Theorem 5.4. *The equation*

$$y^q = x(x-1)^a(x-\lambda')^b,$$

where $1 \le a, b < q$ *and* $|\lambda'| = 1$*, defines a covering of the projective line by a Mumford curve whose topological fundamental group is* Γ_f *as in Lemma 5.3, if and only if*

$$b = q - a \quad \text{and} \quad |\lambda' - 1| < \alpha_p(q,q) = \begin{cases} |1-\zeta_p|^2, & p = q \\ 1, & \text{otherwise.} \end{cases}$$

Here, the number f *is such that* $af \equiv 1 \mod q$.

Proof. The condition $b = q - a$ and $af \equiv 1 \mod q$ on the exponents was found by van Steen using theta functions [17, Proposition 3.2].

The generators for Γ_f from Proposition 3.13 are

$$\gamma_{if} = s^i t s^{-f-i} = \frac{1}{\lambda - 1}\begin{pmatrix} (\lambda\zeta - 1)\zeta^{-f} & \lambda\zeta^i(1-\zeta) \\ \zeta^{-i-f}(\zeta - 1) & \lambda - \zeta \end{pmatrix},$$

where $i = 1, \ldots, q-1$. An automorphism γ of $\mathbb{P}^1$ is hyperbolic if and only if $\frac{|\operatorname{Tr}\gamma|^2}{|\det\gamma|} > 1$. Now,

$$\operatorname{Tr}\gamma_{if} = \frac{(1+\zeta^{1-f})\lambda - (\zeta + \zeta^{-f})}{\lambda - 1}, \qquad \det\gamma_{if} = \zeta^{1-f},$$

therefore,

$$\begin{aligned} & \frac{|\operatorname{Tr}\gamma_{if}|^2}{|\det\gamma_{if}|} > 1 \\ \Leftrightarrow \quad |\lambda - 1| < \; & |(1+\zeta^{1-f})\lambda - (\zeta + \zeta^{-f})| \\ = \; & |1 + \zeta^{1-f} - (\zeta + \zeta^{-f})| \le 1, \end{aligned}$$

where the equality holds, because the difference of the two corresponding terms has norm $|\lambda - 1|$ or less. It follows that, in the case

$$|\lambda - 1| = |1 + \zeta^{1-f} - (\zeta + \zeta^{-f})| = |1-\zeta||1-\zeta^{-f}| = \alpha_p(q,q),$$

the group Γ_f is not discontinuous and therefore does not give rise to a Mumford curve.

Let us assume that $|\lambda - 1| < |1-\zeta|^2$. The isometric circles for γ_{if} and

$$\gamma_{if}^{-1} = s^{f+i} t s^{-i} = \frac{1}{\lambda - 1}\begin{pmatrix} (\lambda\zeta^{-1} - 1)\zeta^f & \lambda\zeta^{f+i}(1-\zeta^{-1}) \\ (\zeta^{-1} - 1)\zeta^{-i} & \lambda - \zeta^{-1} \end{pmatrix}$$

are

$$I_{\gamma_{if}} = \left\{ z \in \mathbb{P}^1 : \left| z - \frac{\zeta - \lambda}{\zeta - 1}\zeta^{i+f} \right| < \frac{|\lambda - 1|}{|\zeta - 1|} \right\},$$

$$I_{\gamma_{if}^{-1}} = \left\{ z \in \mathbb{P}^1 : \left| z - \frac{\zeta^{-1} - \lambda}{1 - \zeta}\zeta^{1-i} \right| < \frac{|\lambda - 1|}{|1 - \zeta|} \right\}.$$

One then sees that

$$I^+_{\gamma_{if}} \cap I^+_{\gamma_{if}^{-1}} = I^+_{\gamma_{if}} \cap I^+_{\gamma_{jf}} = I^+_{\gamma_{if}^{-1}} \cap I^+_{\gamma_{jf}^{-1}} = I^+_{\gamma_{if}} \cap I^+_{\gamma_{jf}^{-1}} = \emptyset$$

for all $i, j = 1, \ldots, q-1$. Therefore, the complement of the union of these open disks is a good fundamental domain for Γ_f in the sense of [7, (4.1.3)]. $\square$

Example 5.5. Specialising the calculations for $q = 2$, one obtains again

$$|\lambda - 1| < |2\lambda + 2| \leq 1 \implies |\lambda - 1| < |2||\lambda + 1| = |2|^2,$$

because indeed $|\lambda + 1| = |1 + 1|$, due to $|\lambda - 1| < 1$.

5.2. **Totally ramified four-point covers.** We assume that $\varphi\colon X \to \mathbb{P}^1$ is of degree m and totally ramified above the four branch points. Let $N_{m,m} = \langle s\rangle * \langle t\rangle$ with s and t of order m.

Theorem 5.6. *Let q be a prime dividing m. Then the bound holds for m if it holds for $m' = \frac{m}{q}$.*

Proof. Assume that the bound holds for m'. We know already that it holds for q. Therefore, the diagram with exact rows and columns (and $\Gamma_{q,q} = \Gamma_f$ as in the preceding subsection)

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Gamma_{q,q} & \longrightarrow & N_{q,q} & \longrightarrow & C_q & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Gamma_{m,m} & \longrightarrow & N_{m,m} & \longrightarrow & C_m & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Gamma_{m',m'} & \longrightarrow & N_{m',m'} & \longrightarrow & C_{m'} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

yields generators for $\Gamma_{m,m}$ which can be examined by the method of isometric circles. Indeed,

$$\Gamma_{m,m} = \langle \Gamma_{q,q}, \zeta_q^i \Gamma_{m',m'} \zeta_q^{-i} \mid i = 0, \dots, q-1\rangle,$$

where ζ_q is a generator of C_q. By assumption, both $\Gamma_{q,q,}$ and $\Gamma_{m',m'}$ are free of rank $q - 1$ and $m' - 1$, respectively. As the right and middle columns are split, also the left column splits. Therefore, $\Gamma_{m,m}$ is free of rank

$$g_{m,m} = (q - 1) + (m' - 1)q = m - 1.$$

The generators obtained in this way from generators of $\Gamma_{q,q}$ and $\Gamma_{m',m'}$ are hyperbolic if and only if the latter are both Schottky groups, which is equivalent to

$$|\lambda - 1| < \min\{\alpha_p(q, q), \alpha_p(m', m')\},$$

by assumption. But then one calculates that the isometric circles of any pairs of different generators of $\Gamma_{m,m}$ and their inverses do not intersect. Thus the bound holds for m. □

Corollary 5.7. *The bound holds for m, if the m-cover is totally ramified.*

Proof. This follows by an iterative application of Theorem 5.6. □

5.3. **Four point covers with arbitrary ramification.** Let $X \to \mathbb{P}^1$ be a cover of degree n ramified above the points $0, 1, \infty, \lambda'$ with $|\lambda'| = 1$ given by the pair (N, Γ) with $N = \langle s \rangle * \langle t \rangle$ and a free normal subgroup Γ realised as the kernel of a surjection $N \to C_n$. Let the orders of s and t be d and e.

5.3.1. *The case $e \mid d$.* In this case, $n = d$, as otherwise X would not be connected. Let ζ be a primitive n-th root of unity, and $f := \frac{n}{e}$. As before, consider the maps

$$\varphi_k \colon N \to C_n, \quad s \mapsto \zeta,\ t \mapsto \left(\zeta^f\right)^k, \qquad (k, e) = 1.$$

The same method by Reidemeister as before yields generators for $\Gamma_k = \ker \varphi_k$

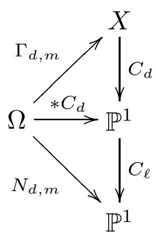
$$B_k := \{\gamma_{ijk} = s^{ik} \gamma_{jk} s^{-ik} \mid i = 1, \ldots, f,\ j = 1, \ldots, e-1\},$$

where

$$\gamma_{jk} = \left(s^{fk}\right)^j\, t \left(s^{fk}\right)^{-j-1}.$$

Theorem 5.8. *The bound holds for m-covers of type (d, m).*

Proof. Let $m = d\ell$ and consider the commutative diagram



The vertical maps $\psi \colon X \to \mathbb{P}^1$ and $\varphi \colon \mathbb{P}^1 \to \mathbb{P}^1$ are cyclic, with a Mumford curve X, and φ is ramified above $\{0, \infty\}$. The branch locus of the horizontal map is $f^{-1}(\{0, \infty, 1, \lambda\})$ which coincides with the branch locus of ψ and is of cardinality $2\ell + 2$. By looking at the corresponding $*$-trees, we see that $T^*_{*C_d}$ has exactly $\ell + 1$ vertices stabilised by C_d: more precisely, from one vertex v on one mirror, there are ℓ paths to the other mirrors, and the pairwise intersection of thes paths is v (in other words, T_{*C_d} is star-shaped with centre v). Hence, $*C_d$ is a free tree product of $\ell + 1$ copies of C_d.

Now, the top triangle with the cyclic cover ψ yields that $\Gamma_{d,m}$ is isomorphic to a free product of ℓ copies of $\Gamma_{d,d}$, which is part of an exact sequence

$$1 \longrightarrow \Gamma_{d,d} \longrightarrow C_d * C_d \longrightarrow C_d \longrightarrow 1$$

However, from the proof of Theorem 5.6 we know that $\Gamma_{d,d}$ is free of rank $d - 1$. For $C_\ell = \langle \zeta \rangle$, this implies that

$$\Gamma_{d,m} = \langle \zeta^i \Gamma_{d,d} \zeta^{-i} \mid i = 0, \ldots, \ell - 1 \rangle$$

is free of rank

$$g_{d,d\ell} = \ell(d-1).$$

By Corollary 5.7, the bound holds for ψ. The section $T^*_{N_{d,m}} \to T^*_{*C_d}$ being isometric implies that the bound holds for $\varphi \circ \psi$. $\square$

5.3.2. *The case* $(d,e)=1$. In the case that $(d,e)=1$, it follows that necessarily $n=ed$. Consider the maps

$$\varphi_{k\ell}\colon N\to C_n,\quad s\mapsto\zeta^{ek},\ t\mapsto\zeta^{d\ell},$$

where $(k,e)=1$ and $(\ell,d)=1$. Let $\sigma:=s^e$, $\tau:=t^d$ and

$$B_{k\ell}:=\{\gamma_{ij}:=\sigma^{-i}\tau^{-j}\sigma^i\tau^j\mid i=1,\dots,d-1,\ j=1,\dots,e-1\}.$$

Proposition 5.9. $\Gamma_{k\ell}:=\ker\varphi_{k\ell}$ *is free of rank* $(e-1)(d-1)$.

Proof. This follows from a similar Riemann-Hurwitz argument as in the proof of Proposition 3.13. □

Now assume that $(p,d)=1$.

Theorem 5.10. *The equation*

$$y^n=x^a(x-1)^b(x-\lambda')^{n-b},$$

where $1\le a<n$ *is of order* e *mod* n, $1\le b<n$ *of order* d *mod* n, $(d,e)=1$ *and* $|\lambda'|=1$ *defines a Mumford curve covering* $\mathbb{P}^1$, *if and only if* $|\lambda'-1|<\alpha_p(1,e)$.

Proof. From

$$\sigma^i\tau^j=\frac{1}{\lambda-1}\begin{pmatrix}(\lambda\zeta^{dj}-1)\zeta^{ei} & \lambda(1-\zeta^{dj}\zeta^{ei})\\ \zeta^{dj}-1 & \lambda-\zeta^{dj}\end{pmatrix},$$

$$\sigma^{-i}\tau^{-j}=\frac{1}{\lambda-1}\begin{pmatrix}(\lambda\zeta^{-dj}-1)\zeta^{-ei} & \lambda(1-\zeta^{-dj}\zeta^{-ei})\\ \zeta^{-dj}-1 & \lambda-\zeta^{-dj}\end{pmatrix}$$

we calculate

$$\gamma_{ij}:=\sigma^{-i}\tau^{-j}\sigma^i\tau^j=\frac{1}{(\lambda-1)^2}\begin{pmatrix}a_{ij} & b_{ij}\\ c_{ij} & d_{ij}\end{pmatrix}$$

with

$$\begin{aligned}
a_{ij}&=(\lambda\zeta^{dj}-1)(\lambda\zeta^{-dj}-1)-\lambda\zeta^{-ei}(\zeta^{dj}-1)(\zeta^{-dj}-1),\\
b_{ij}&=\lambda(\lambda\zeta^{-dj}-1)(1-\zeta^{dj})+\lambda\zeta^{-ei}(1-\zeta^{-dj})(\lambda-\zeta^{dj}),\\
c_{ij}&=(\zeta^{-dj}-1)(\lambda\zeta^{dj}-1)\zeta^{ei}+(\lambda-\zeta^{-dj})(\zeta^{dj}-1),\\
d_{ij}&=\lambda\zeta^{ei}(\zeta^{-dj}-1)(1-\zeta^{dj})+(\lambda-\zeta^{-dj})(\lambda-\zeta^{dj}).
\end{aligned}$$

By definition, $\det\gamma_{ij}=1$. The trace of γ_{ij} is

$$\operatorname{Tr}\gamma_{ij}=\frac{2(\lambda-\zeta^{-dj})(\lambda-\zeta^{dj})-\lambda(\zeta^{-ei}+\zeta^{ei})(1-\zeta^{dj})(1-\zeta^{-dj})}{(\lambda-1)^2}.$$

Thus the condition for hyperbolicity of γ_{ij} is

$$|\lambda-1|^2<|2(\lambda-\zeta^{-dj})(\lambda-\zeta^{dj})-\lambda(\zeta^{-ei}+\zeta^{ei})(1-\zeta^{dj})(1-\zeta^{-dj})|.$$

Set

$$\varepsilon_{ij}:=(1-\zeta^{dj})(1-\zeta^{-dj})(1-\zeta^{ei})(1-\zeta^{-ei}),$$

and notice that

$$\zeta^{ei}+\zeta^{-ei}=2-(1-\zeta^{ei})(1-\zeta^{-ei}).$$

Therefore, the right hand side of the inequality equals

$$\begin{aligned}
&|2((\lambda-\zeta^{-dj})(\lambda-\zeta^{dj})-\lambda(1-\zeta^{dj})(1-\zeta^{-dj}))+\lambda\varepsilon_{ij}| \\
=&|2(\lambda^2-\lambda(\zeta^{dj}+\zeta^{-dj})+1-2\lambda+\lambda(\zeta^{dj}+\zeta^{-dj}))+\lambda\varepsilon_{ij}| \\
=&|2(\lambda-1)^2+\lambda\varepsilon_{ij}| \\
=&|\lambda\varepsilon_{ij}|=|\varepsilon_{ij}|,
\end{aligned}$$

where the first equality in the last line holds true, because $|2(\lambda-1)^2|\leq|\lambda-1|^2$.

In a similar way we obtain

$$\gamma_{ij}^{-1}=\tau^{-j}\sigma^{-i}\tau^{j}\sigma^{i}=\frac{1}{\lambda-1}\begin{pmatrix}a'_{ij} & b'_{ij}\\ c'_{ij} & d'_{ij}\end{pmatrix},$$

with

$$\begin{aligned}
a'_{ij}&=(\lambda\zeta^{dj}-1)(\lambda\zeta^{-dj}-1)-\lambda\zeta^{ei}(\zeta^{dj}-1)(\zeta^{-dj}-1),\\
b'_{ij}&=\lambda\zeta_{-ei}(\lambda\zeta^{-dj}-1)(1-\zeta^{dj})+\lambda(1-\zeta^{-dj})(\lambda-\zeta^{dj}),\\
c'_{ij}&=(\zeta^{-dj}-1)(\lambda\zeta^{dj}-1)+\zeta^{ei}(\lambda-\zeta^{-dj})(\zeta^{dj}-1),\\
d'_{ij}&=\lambda\zeta^{-ei}(\zeta^{-dj}-1)(1-\zeta^{dj})+(\lambda-\zeta^{-dj})(\lambda-\zeta^{dj}).
\end{aligned}$$

The isometric circles are

$$\begin{aligned}
I_{\gamma_{ij}}&=\left\{z\in\mathbb{P}^1:\left|z+\frac{d_{ij}}{c_{ij}}\right|<\frac{|\lambda-1|^2}{|c_{ij}|}\right\}\\
I_{\gamma_{ij}^{-1}}&=\left\{z\in\mathbb{P}^1:\left|z+\frac{d'_{ij}}{c'_{ij}}\right|<\frac{|\lambda-1|^2}{|c'_{ij}|}\right\}.
\end{aligned}$$

They do not intersect pairwise, if and only if

$$|\lambda-1|^2<\min\left\{|d_{ij}-d'_{ij}|,\ |d_{ij}-d_{i'j'}|,\ |d'_{ij}-d'_{i'j'}|\right\},$$

where $i,i'=1,\dots,d-1$ and $j,j'=1,\dots,e-1$ are such that the set to be minimised does not contain zero.

Rewrite d_{ij} as

$$\begin{aligned}
d_{ij}&=(\lambda-\zeta^{-dj})(\lambda-\zeta^{dj})-\lambda\zeta^{ei}(\zeta^{-dj}-1)(\zeta^{dj}-1)\\
&=\lambda^2-\lambda(\zeta^{-dj}+\zeta^{dj})+1-\lambda\zeta^{ei}\left(2-(\zeta^{dj}+\zeta^{-dj})\right)-2\lambda+2\lambda\\
&=(\lambda-1)^2+2\lambda(1-\zeta^{ei})+\lambda(\zeta^{dj}-\zeta^{dj})(\zeta^{ei}-1)\\
&=(\lambda-1)^2+\lambda(1-\zeta^{ei})\left(2-(\zeta^{dj}+\zeta^{-dj})\right)\\
&=(\lambda-1)^2+\lambda(1-\zeta^{ei})(1-\zeta^{dj})(1-\zeta^{-dj}),
\end{aligned}$$

and, similarly, d'_{ij} as

$$d'_{ij}=(\lambda-1)^2+\lambda(1-\zeta^{-ei})(1-\zeta^{dj})(1-\zeta^{-dj}),$$

and set $e=p^r\ell$, $(p,\ell)=1$. Then the minimum is attained for $j=j'=p^{r-1}\ell$ and takes the value

$$|d_{ij}-d_{i'j}|=|\lambda|\cdot|\zeta^{ei'}-\zeta^{ei}|\cdot|(1-\zeta^{dj})(1-\zeta^{-dj})|=|1-\zeta_p|^2,$$

since we assumed $(d,p)=1$. □

5.3.3. *The case $d \nmid e$ and $e \nmid d$.* In the case $d \nmid e$ and $e \nmid d$, we have

$$n = \operatorname{lcm}(d,e) \quad \text{and} \quad \ell = \gcd(d,e).$$

Theorem 5.11. *The equation*

$$y^n = x^a(x-1)^b(x-\lambda')^{n-b}$$

where $1 \leq a < n$ is of order e mod n, $1 \leq b < n$ of order d mod n, and $|\lambda'| = 1$ defines a Mumford curve covering $\mathbb{P}^1$, if and only if

$$|\lambda' - 1| < \alpha_p(d,e).$$

Proof. Let $d' := \frac{d}{\ell}$, $e' := \frac{e}{\ell}$, $m := d'e'$ and consider the diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Gamma_{d',e'} & \longrightarrow & N_{e',d'} & \longrightarrow & C_m & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Gamma_{d,e} & \longrightarrow & N_{e,d} & \longrightarrow & C_n & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Gamma_{\ell,\ell} & \longrightarrow & N_{\ell,\ell} & \longrightarrow & C_\ell & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

with exact rows and columns, where $N_{a,b} = C_a * C_b$, and the arrows $N_{d',e'} \to C_m$ and $N_{\ell,\ell} \to C_\ell$ are as in Section 5.3.2 and Lemma 5.3, respectively. Thus, $\Gamma_{d',e'}$ and $\Gamma_{\ell,\ell}$ are free of ranks $(d'-1)(e'-1)$ and $\ell-1$, respectively. From the diagram, it follows that $\Gamma_{d,e}$ is generated by $\Gamma_{e',d'}$ and the C_m-orbits of $\Gamma_{\ell,\ell}$, where C_m acts by conjugation with the powers of some primitive m-th root of unity contained in $N_{d,e}$. As the right and the middle columns are split, also the left column splits. Therefore, $\Gamma_{d,e}$ is free and is of rank

$$g = (d'-1)(e'-1) + (\ell-1)\cdot m,$$

and we can construct in an obvious way explicit generators for $\Gamma_{d,e}$ from the generating systems of $\Gamma_{d',e'}$ and $\Gamma_{\ell,\ell}$ given earlier. Again one checks that these generators yield a Schottky group if and only if

$$|1-\lambda| < \alpha_p(d,e).$$

□

Remark 5.12. *We are convinced that one can refine the method in* [17] *in order to relate to arbitrary m-covers the precise Schottky group, as constructed here.*

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