

RIEMANN EXISTENCE THEOREMS OF MUMFORD TYPE

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ABSTRACT. Riemann Existence Theorems for Galois covers of Mumford curves by Mumford curves are stated and proven. As an application, all finite groups are realised as full automorphism groups of Mumford curves in characteristic zero.

1. INTRODUCTION

The classical Riemann Existence Theorem (RET) states that a finite group G can be realised as a deck transformation group of a G -cover of the Riemannian sphere by a compact Riemann surface, if and only if G is generated by some elements whose product is one. This amounts to finding a \mathbb{C} -rational point of a Hurwitz space.

If in the situation of p -adic geometry we consider Mumford curves to be the analogue of Riemann surfaces, the translation ad literam of the RET is no longer valid. The purpose of this article is to find a correct translation of the RET in the characteristic zero case. This means that we relate the non-emptiness of Mumford-Hurwitz spaces constructed in [6] to certain generating systems with the right properties, i.e. it is exactly these properties we will explain in this article.

For this, the notion of Mumford orbifold becomes crucial. These are Mumford curves covered by Mumford curves, and to such a cover φ belongs a certain graph of groups encoding the ramification behaviour of φ which in the tree-case was called **-tree* by F. Kato who studied them first. In the general case, we call it a **-graph*. Exploiting its structure gives the translation we seek. We call the result *virtual RET*, because of the interpretation of the stabilisers of vertices in **-graphs* as 'virtual' ramification—something that the orbifold itself does not 'see'.

In the special case that all edge groups are trivial, we recover the so-called *Harbater-Mumford components* of Hurwitz spaces considered by Fried in [9]. This leads to another proof of Harbater's result that every finite group can be realised as the Galois group of a cover $X \rightarrow \mathbb{P}_{\mathbb{C}_p}^1$ of the complex p -adic projective line in characteristic zero by a Mumford curve X [10]. He does the job with 'mock covers' which are totally degenerate covers over the residue field, i.e. they lift to covers by Mumford curves.

Until now, it was an open question, if every finite group can be realised as the full automorphism group of a Mumford curve in characteristic zero. Our techniques yield a positive answer.

The setup for this paper is p -adic geometry of characteristic zero, that is we are working over the field \mathbb{C}_p obtained by completing an algebraic closure of the field \mathbb{Q}_p of rational p -adic numbers.

2. HURWITZ SPACES

2.1. Covers of projective curves. Let G be a finite group of order n and $\varphi: X \xrightarrow{G} Y$ a cover of a non-singular projective irreducible algebraic curve Y of genus g over a field K of characteristic zero, ramified above r distinct K -rational points η_1, \dots, η_r

with orders e_1, \dots, e_r . It is given by a surjective map of topological groups $\psi: \pi_1^{\text{alg}}(Y \setminus \{\eta_1, \dots, \eta_r\}, y) \rightarrow \text{Aut } \varphi \cong G$.

Taking topological generators $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma_1, \gamma_r$ of $\pi_1^{\text{alg}}(Y \setminus \{\eta_1, \dots, \eta_r\}, y)$ with the relation

$$\prod_j [\alpha_j, \beta_j] \prod_i \gamma_i = 1,$$

where $[\alpha, \beta] = \alpha^{-1}\beta^{-1}\alpha\beta$, we obtain a generating system

$$a_j := \psi(\alpha_j), b_j := \psi(\beta_j), g_i := \psi(\gamma_i)$$

of G with the same relation, called *genus g generating system*. The conjugacy classes C_i of the g_i define the *ramification type* $\mathbf{C} := (C_1, \dots, C_r)$ of the cover φ .

If $K = \mathbb{C}$, then γ_i can be viewed as a closed topological path in $Y \setminus \{\eta_1, \dots, \eta_r\}$ separating η_i from the other branch points, and lifting the path to X gives a deck transformation $g_i \in \text{Aut } \varphi$ which permutes the fibre $\varphi^{-1}(y)$ (y is not assumed to be a branch point). Conjugation means changing the genus g generating system of G .

Let $\mathcal{M}_{g,r}$ be the moduli stack of smooth r -pointed curves of genus g , and $\mathcal{X} \rightarrow \mathcal{M}_{g,r}$ the universal object. For a finite group G of order n with its regular representation $G \rightarrow S_n$ into the symmetric group, there is, by [22, Remark 4.1.3], a smooth algebraic stack $\mathcal{H}_{\mathcal{X}}^G(G)$ whose coarse moduli scheme $H_g(G, r)$ is finitely étale over the moduli scheme $M_{g,r}$ of $\mathcal{M}_{g,r}$. This moduli space parametrises equivalence classes of Galois covers $Y \rightarrow X$ of smooth curves of genus g ramified above r points together with an isomorphism $\sigma: G \rightarrow \text{Aut}(Y/X)$. Two pairs $(Y \xrightarrow{G} X, \sigma)$ and $(Y' \xrightarrow{G} X', \sigma')$ are equivalent, if there is a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sigma} & X' \end{array}$$

with horizontal isomorphisms, and such that $f \circ \sigma(g) \circ f^{-1} = \sigma'(g)$ for all $g \in G$.

Define $\mathcal{H}_{\mathcal{X}}(G, \mathbf{C})$ to be the algebraic substack of $\mathcal{H}_{\mathcal{X}}^G(G)$ of covers with ramification type \mathbf{C} over any irreducible scheme whose function field is of characteristic zero, and let $H_g(G, \mathbf{C})$ be its coarse moduli scheme. It is a quasi-projective scheme over $\mathbb{Z}[\frac{1}{n}]$ [5]. The space of \mathbb{C} -rational points $H_g(G, \mathbf{C})(\mathbb{C})$ is non-empty if and only if G has a genus g generating system. This is in the genus zero case nothing but the Riemann Existence Theorem, the starting point of our observations from Section 3 on.

Definition 2.1. *The spaces*

$$\mathbf{H}_g(G, \mathbf{C}) := (H_g(G, \mathbf{C}) \otimes_{\mathbb{Z}[\frac{1}{n}]} \mathbb{C}_p)^{\text{an}} \quad \text{and} \quad \mathbf{H}_g(G, r) := (H_g(G, r) \otimes_{\mathbb{Z}[\frac{1}{n}]} \mathbb{C}_p)^{\text{an}},$$

where the superscript ^{an} means analytification of schemes, are called p -adic Hurwitz spaces.

We will also consider the intermediate Hurwitz spaces $\mathbf{H}_g(G, \mathbf{e})$ of covers with fixed list of ramification orders $\mathbf{e} = (e_1, \dots, e_r)$ which we call the *signature* of the covers parametrised by the Hurwitz space.

2.2. *-graphs. If not stated otherwise, K will always denote a finite extension field of the p -adic field \mathbb{Q}_p which is supposed to contain all ramification points of covers. This is justifiable, as all equivalence classes of covers of curves defined over \mathbb{C}_p contain in fact a representative defined over some finite extension field of \mathbb{Q}_p .

In the non-Archimedean setting one faces the problem that there are not enough topological coverings, e.g. it is a well known fact that all analytic subsets of \mathbb{P}^1 or all curves with tree-like reduction are topologically simply connected. For this reason, more general covers are often taken into consideration.

Definition 2.2 (De Jong). *An analytic map $f: S' \rightarrow S$, with S a connected p -adic manifold, is an étale covering, if S is covered by open subsets U such that $f^{-1}(U) = \coprod V_j$ and $f|_{V_j}$ is a finite étale morphism.*

By a p -adic manifold we mean a smooth paracompact strictly \mathbb{C}_p -analytic space in the sense of [3]. Finite étale morphisms are examples of étale covers. If the $f|_{V_j}$ in Definition 2.2 are all isomorphisms, then we have *topological covers*. These correspond to locally constant sheaves of sets on S . In [8, Lemma 2.2] it is shown that étale covers $f: S' \rightarrow S$ are étale and separated morphisms of analytic spaces; if S is paracompact then also is S' . Thus if f is an étale cover of a p -adic manifold, then S' is also a p -adic manifold. According to [4], such spaces are locally compact, locally arcwise connected and locally contractible. This means that the usual theory of universal coverings and topological fundamental groups applies.

For an étale cover $S' \rightarrow S$, the étale quotient sheaf S'/R on S by an equivalence relation $R \subseteq S' \times_S S'$ is representable by an étale cover $S'' \rightarrow S$, if R is a union of connected components [8]. In this case, we call $S'' \rightarrow S$ a *quotient* of $S' \rightarrow S$. This leads to André's notion of *tempered étale covers*.

Definition 2.3 (André). *A tempered cover $S' \rightarrow S$ is a quotient of a composite étale cover $T' \rightarrow T \rightarrow S$, where $T' \rightarrow T$ is a topological covering and $T \rightarrow S$ a finite étale cover.*

Fixing the ramification orders in covers yields the notion of orbifold which can quickly be defined as follows: a (one-dimensional uniformisable) *orbifold* $\mathcal{S} = (S, (\zeta_i, e_i))$ consists of a p -adic manifold S of dimension one and finitely many points ζ_i allowing a Galois cover $\Omega \rightarrow S$ (a *global chart*) which is tempered outside $\{\zeta_i\}$ and of ramification orders e_i above ζ_i .

Definition 2.4. *A Mumford orbifold is a one-dimensional orbifold \mathcal{S} having a global chart $\varphi: X \rightarrow S$ with a Mumford curve X (where the projective line and Tate curves are also considered as Mumford curves). If $\mathcal{S} = (\mathbb{P}^1, (0, e_0), (1, e_1), (\infty, e_\infty))$, then \mathcal{S} will be called a Mumford-Schwarz orbifold.*

Let \mathfrak{M}_g be the moduli space defined over \mathbb{C}_p of Mumford curves of genus g . It is an analytic subspace of the moduli space $M_g \otimes \mathbb{C}_p$ of all genus g curves.

Definition 2.5. *The analytic space defined by the pullback via the cartesian diagram*

$$\begin{array}{ccc} \mathfrak{H}_g(G, \mathbf{e}) & \longrightarrow & \mathbf{H}_g(G, \mathbf{e}) \\ \downarrow & & \downarrow \\ \mathfrak{M}_h & \longrightarrow & M_h \otimes \mathbb{C}_p \end{array}$$

where the vertical arrows map a cover to the upper curve, and h is given by the Riemann-Hurwitz formula, is called a Mumford-Hurwitz space.

Mumford-Hurwitz spaces parametrise G -covers of Mumford orbifolds of genus g with fixed lists of ramification orders $\mathbf{e} = (e_1, \dots, e_r)$. In [6] we constructed them using Herrlich's non-archimedean Teichmüller spaces from [14], and also André's orbifold fundamental group $\pi_1^{\text{orb}}(\mathcal{S}, \bar{s})$ from [1].

Lemma 2.6. *The Mumford-Hurwitz space $\mathfrak{H}_g(G, \mathbf{e})$ is open in $\mathbf{H}_g(G, \mathbf{e})$.*

Proof. This follows from the well known fact that \mathfrak{M}_h is open in $M_h \otimes \mathbb{C}_p$ (h as in Definition 2.5), which in turn holds because the fibres of the reduction map into the boundary of the moduli space of curves over $\overline{\mathbb{F}}_p$ are open in $\mathbf{H}_g(G, \mathbf{e})$. \square

From Definition 2.4 it follows that if \mathcal{S} is a Mumford orbifold, then S is a Mumford curve (e.g. [6, Lemma 4.2]). Let $\Omega \rightarrow X$ be the topological universal cover of X . The space Ω is an analytic subdomain of the p -adic projective line, and it fits into a commuting diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{F_g} & X \\ & \searrow N & \downarrow G \\ & & \mathcal{S} \end{array}$$

where G is a finite group and F_g is the free group on $g = \text{genus}(X)$ generators. In fact, Ω is the complement in \mathbb{P}^1 of the limit points of N acting discontinuously as a subgroup of $\text{PGL}_2(\mathbb{C}_p)$. According to [2, 4.1.7], there is an induced action of the discrete group N on the skeleton $\Delta(\Omega)$, a tree whose ends correspond bijectively to the set \mathcal{L} of limit points of N , and there is a pure affinoid covering \mathfrak{U} of S such that the quotient $\Delta(\Omega)/N$ is isomorphic to the intersection graph of the analytic reduction $\Delta_{\mathfrak{U}}(S)$ of S with respect to \mathfrak{U} . The graph of groups $\mathfrak{G}(N) := (\Delta_{\mathfrak{U}}(S), N_{\bullet})$ contains some information on the cover $X \xrightarrow{G} \mathcal{S}$: since the topological fundamental group of X is a free group, all vertex groups of $\mathfrak{G}(N)$ are contained in G . They coincide with the stabilisers under the action of G on the reduction graph $\Delta_{\varphi^{-1}(\mathfrak{U})}(X)$ of X . But there is no direct information on the ramification of the cover φ itself. As, according to [11, Satz 5], the ramification points of φ are exactly the F_g -orbits of fixed points in Ω of elliptic transformations of N , Kato remedies this problem by considering the complement Ω^* of the set \mathcal{L}^* which is the union of \mathcal{L} and the set of fixed points of elements of finite order in N . As N acts also on $\Omega^* \subseteq \Omega$, the graph of groups $\mathfrak{G}^* := (\Delta(\Omega^*)/N, N_{\bullet})$ is well-defined. Whereas \mathfrak{G} is a finite graph of groups, \mathfrak{G}^* is an infinite graph with finitely many cusps which correspond bijectively to the branch points of φ and whose stabilisers coincide with the decomposition groups [16, Proposition 2.2]. However, both \mathfrak{G} and \mathfrak{G}^* have fundamental groups isomorphic to N . As all vertex groups are finite, this means by [18] that there is a finite number of *admissible* edge contractions or *slides* which turns the finite part \mathfrak{G}' of \mathfrak{G}^* (obtained by contracting the cusps) to \mathfrak{G} . Here, we say that a contraction of an edge e in a graph of groups $(\mathfrak{G}, N_{\bullet})$ is *admissible*, if N_e is isomorphic to either $N_{o(e)}$ or $N_{t(e)}$, where $o(e)$ is the origin vertex of e and $t(e)$ the terminal vertex of e and $o(e) \neq t(e)$. A *slide* means replacing an embedding $\alpha: N_e \rightarrow N_v$ (with $v = o(e)$ or $v = t(e)$) by $c_g \circ \alpha$, where c_g is conjugation by $g \in N_v$, but leaving the underlying graph \mathfrak{G} unchanged.

Definition 2.7. \mathfrak{G}^* is called the **-graph* of φ .

If \mathfrak{G}^* is a tree, then we call it a **-tree*, of course. If $X \xrightarrow{G} \mathcal{S}$ is a finite global chart of a Mumford orbifold with at most three branch points, then the corresponding **-tree* is called an *elementary *-tree*. G is then a finite group of projective linear transformations.

2.3. The dimension of Mumford-Hurwitz spaces. By Lemma 2.6, the dimensionality of the Mumford-Hurwitz space $\mathfrak{H}_g(G, \mathbf{e})$ is already settled: it is $3g - 3 + m$, where m is the number of branch points of covers belonging to $\mathfrak{H}_g(G, \mathbf{e})$. The goal of this subsection is to relate the number of branch points to the structure of the graph of groups associated to such a cover.

In fact, the number of branch points does not depend on the particular choice of the discrete embedding of a finitely generated group into $\mathrm{PGL}_2(K)$:

Proposition 2.8 (with H. Voskuil). *Let $\tau: N \hookrightarrow \mathrm{PGL}_2(K)$ be a discrete embedding of a finite tree of finite groups which are amalgamated along non-trivial finite groups. Then the number $m = \#\mathrm{Ends}(T_N^*)$ does not depend on the particular choice of τ . Moreover, the stabilisers of ends of T_N^* (as abstract groups) do not depend on τ .*

Proof. Consider an end \mathcal{E} of the tree $\mathcal{T} := \mathcal{T}_{\tau(N)}$ which is contained in the $*$ -tree $\mathcal{T}_{\tau(N)}^*$. We first prove that stabilisers of vertices along \mathcal{E} do not form a descending chain. More precisely, if we take any half line beginning at vertex v_0 and whose equivalence class is \mathcal{E} , then the stabilisers N_0, N_1, \dots of the consecutive vertices v_0, v_1, \dots does not descend. Otherwise, the sequence $N_0 \supseteq N_1 \supseteq \dots$ would become stationary from a vertex v_i on with smallest group $N_{v_i} = C_\ell$ cyclic. By assumption, N_{v_i} must be non-trivial. This means that there is an element $\alpha \in N \setminus \{1\}$ of finite order fixing infinitely many vertices of \mathcal{T} . By [11, Satz 6], α commutes with an hyperbolic element $\gamma \in \tau(N) \setminus \{1\}$. But then the quotient graph \mathcal{T}/N contains a loop, which is impossible.

It follows from the above that if one takes a contraction $\tilde{\mathcal{T}}$ of \mathcal{T} such that for all maximal finite subgroups $H \subseteq N$ there is a vertex v in $\tilde{\mathcal{T}}$ stabilised by H , then to each end of $\tilde{\mathcal{T}}$ there corresponds a unique end of \mathcal{T} . Moreover, stabilisers of ends do not change under this correspondence. Since for any discrete embedding $\tau': N \rightarrow \mathrm{PGL}_2(K)$ the tree $\mathcal{T}' = \mathcal{T}_{\tau'(N)}$ can be contracted to $\tilde{\mathcal{T}}$, one concludes (using again [11, Satz 6]) that for any finite-order element $\alpha \in N \setminus \{1\}$ the assertion

$$\text{fixed points of } \tau(\alpha) \text{ are regular} \iff \text{fixed points of } \tau'(\alpha) \text{ are regular}$$

holds. This proves the proposition, as the ends of $T_{\tau(N)}^*$ are the N -orbits of regular fixed points in \mathbb{P}_K^1 of elements of $N \setminus \{1\}$ of finite order. \square

Definition 2.9. *A graph of groups $(\mathfrak{G}, N_\bullet)$ with finite vertex groups is called stable, if for each vertex v of valency ≤ 2 and each edge e with origin v the inclusion $N_e \rightarrow N_v$ maps N_e to a proper subgroup of N_v .*

AMALGAMIFICATION

Let N be a finitely generated group, and K a complete non-archimedean valued field. Denote further $T(N)$ the space of representations $\tau: N \rightarrow \mathrm{PGL}_2(K)$ such that τ is injective, and $\tau(N)$ acts discontinuously and does not contain parabolic elements. $T(N)$ is invariant under action of $\mathrm{PGL}_2(K)$, and the quotient $\bar{T}(N)$ is called the *Teichmüller space* for N over K . It is a fine moduli space, and the quotient $\mathfrak{M}(N) := \bar{T}(N)/\mathrm{Out}(N)$, where the outer automorphism group $\mathrm{Out}(N)$ acts discontinuously, parametrises N -uniformisable Mumford curves [15].

Mumford curves uniformised by groups of a certain type allow some control over the branch locus of the corresponding covers, as the following will reveal.

Definition 2.10. *We say that a group which is a free product of a free group of finite rank with a free amalgamated product of finitely many finite groups is of type **Am**.*

Let $N = G_1 *_H G_2$ be an amalgam of finitely generated groups, $\tau \in T(N)$, $\tau_i := \tau|_{G_i}$ and $\tau_H := \tau|_H$. Denote the corresponding sets of ordinary points in \mathbb{P}^1 as Ω , Ω_i and

Ω_H . We clearly have $\Omega_i \subseteq \Omega_H$ and $\Omega \subseteq \Omega_1 \cap \Omega_2$, where in general equalities do not hold. We are interested in the branch loci of the covers

$$f: \Omega \rightarrow \Omega/N, \quad f_i: \Omega_i \rightarrow \Omega_i/G_i, \quad f_H: \Omega_H \rightarrow \Omega_H/H$$

induced by τ . Denote the branch loci of the above covers with $br(f)$, $br(f_i)$ and $br(f_H)$.

Lemma 2.11. *If $N = G_1 *_H G_2$ is of type **Am**, then*

$$br(f_H) = br(f_H|_\Omega) \quad \text{and} \quad br(f_i) = br(f_i|_\Omega).$$

Proof. Let $x \in br(f_H)$ resp. $\in br(f_i)$. It is the H -orbit resp. G_i -orbit of a point $z \in \Omega_H$ resp. $\in \Omega_i$ fixed by an element α of finite order in $H \setminus \{1\}$ resp. in $G_i \setminus \{1\}$. If x were not a branch point of the restriction to Ω , then there would exist a z as above, but outside Ω . By [11, Satz 6], this means that there exists a hyperbolic $\gamma \in N$ which commutes with α . But then a stable graph \mathfrak{G} of groups with fundamental group N contains an edge which is a loop and is stabilised by α . This implies that N cannot be of type **Am**. \square

The calculation of the dimension of the spaces $T(N)$ is reduced by Herrlich to considering the case of type **Am** by constructing a finite étale map $T(N) \rightarrow T(N')$ with N' of type **Am**.

Lemma 2.12 (Herrlich). *If $T(N) \neq \emptyset$ then there is a subgroup $N' \subseteq N$ which is of type **Am** and the canonical map $T(N) \rightarrow T(N')$ is finite and unramified.*

Proof. Let \mathfrak{G} be a stable graph of groups with fundamental group N . Following the proof of [15, Lemma 14], we need to consider only the case, where the underlying graph Q of \mathfrak{G} is such that for every maximal subtree P the complement $Q \setminus P$ contains an edge e with nontrivial stabiliser N_e . There are only two possible cases:

- (Am 1) The edge e is a loop, and $N_e = N_{o(e)}$ is cyclic. In this case, let \mathfrak{G}' be the graph of groups obtained from \mathfrak{G} by replacing $N(e)$ and its images under all embeddings of edge groups by the trivial group. Then N is generated by the fundamental group N' of \mathfrak{G}' and an element α commuting with the element γ_e belonging to the edge e . So, any $\tau \in T(N')$ can be extended in finitely many ways to an element $\tilde{\tau} \in T(N)$ such that $\tilde{\tau}(\alpha)$ is non-trivial of finite order. This gives a finite and unramified map $T(N) \rightarrow T(N')$.
- (Am 2) The vertex group $N_{o(e)} =: N_1$ is not cyclic. In this case, \mathfrak{G} is replaced by a graph of groups \mathfrak{G}' by giving e a new vertex v as origin, and setting $N'_v = N_1$. Viewing the fundamental group N' of \mathfrak{G}' as a subgroup of N , we have in the bigger group the relation $N'_v = \gamma_e N_1 \gamma_e^{-1}$, which means that the induced map $T(N) \rightarrow T(N')$ is again finite and unramified.

\square

Definition 2.13. *The group N' of Lemma 2.12 is called an amalgamification of N .*

Theorem 1. *Let N be a finitely generated group containing a free subgroup of rank 2, and \mathfrak{G} a stable graph of groups whose fundamental group is N . Let further $\tau \in T(N)$. If $\Omega \subseteq \mathbb{P}^1$ is the set of ordinary points of $\tau(N)$, then the cover $\Omega \rightarrow \Omega/N$ of the Mumford curve Ω/N has exactly*

$$n = 2(C - c) + 3(D - d)$$

branch points, where C resp. D denotes the number of non-trivial cyclic resp. non-cyclic vertex groups and c resp. d the number of non-trivial cyclic resp. non-cyclic edge groups of \mathfrak{G} .

Proof. We will first show how n does change if we replace N by an amalgamification N' . We use the notations from the proof of Lemma 2.12

In the case (Am 1), $\tau(\alpha)$ has the same fixed points as the hyperbolic $\tau(\gamma)$, which means that these are necessarily limit points of $\tau(N)$ but not of $\tau(N')$. Therefore the induced covers

$$f: \Omega \rightarrow \Omega/N \quad \text{and} \quad f': \Omega' \rightarrow \Omega'/N',$$

where Ω resp. Ω' are the sets of ordinary points in \mathbb{P}^1 of $\tilde{\tau}(N)$ resp. $\tau(N')$, have a difference of 2 in the number of branch points.

In the case (Am 2), N is generated by N' , γ_e and N_1 . As the latter group is conjugated to N'_v , the number of branch points goes up by 3.

Let now $N = G_1 *_H G_2$ be of type **Am**. Let f correspond to a representation $\tau \in T(N)$, and f_i resp. f_H the covers corresponding to $\tau|_{G_i}$ resp. $\tau|_H$. By Lemma 2.11, the commutative square

$$\begin{array}{ccc} \Omega/H & \longrightarrow & \Omega/G_1 \\ \downarrow & & \downarrow \\ \Omega/G_2 & \longrightarrow & \Omega/N \end{array}$$

induced by τ yields a commutative square of maps between the branch loci

$$(1) \quad \begin{array}{ccc} br(f_H) & \longrightarrow & br(f_1) \\ \downarrow & & \downarrow \varphi_1 \\ br(f_2) & \xrightarrow{\varphi_2} & br(f) \end{array}$$

This yields the inequality

$$n \geq \#br(f) =: m, \quad \text{i.e.} \quad 3g - 3 + n \geq 3g - 3 + m,$$

because $br(f)$ is the union of the images of φ_1 and φ_2 : a branch point of f is the N -orbit of a fixed point in Ω of an element $\alpha \in N$ of finite order which lies in a conjugate of G_1 or G_2 .

Let now F_h be a finitely generated free normal subgroup in N of finite index and rank ≥ 2 . The moduli space $\mathfrak{M}(N, F_h) = \bar{T}(N)/\text{Out}_{F_h}(N)$, where $\text{Out}_{F_h}(N)$ is the group of F_h -invariant outer automorphisms of N , parametrises commuting diagrams

$$\begin{array}{ccc} \Omega & \longrightarrow & \Omega/F_h \\ & \searrow & \downarrow \\ & & \Omega/N \end{array}$$

[13]. By Proposition 2.8 it follows that $\mathfrak{M}(N, F_h)$ embeds into $\mathfrak{H}_g(N/F_h, m)$. This Mumford-Hurwitz space has dimension $3g - 3 + m$ by Lemma 2.6, therefore, we have $\dim \mathfrak{M}(N, F_h) = 3g - 3 + n \leq 3g - 3 + m$, in other words: $n \leq m$. \square

Corollary 2.14. *If $G = G_1 *_H G_2$ is of type **Am**, then*

$$br(f) = br(f_1) \#_{br(f_H)} br(f_2),$$

where $\#$ denotes the pushout (here: of sets).

Proof. In the proof of Theorem 1 we have seen that $br(f)$ is the union of the images of φ_1 and φ_2 in Diagram (1), so there is a surjective map from the pushout to $br(f)$. The equality of the finite cardinalities of $br(f)$ and the pushout show that the two sets are equal. \square

3. THE MUMFORD CURVE LOCUS

3.1. The Harbater-Mumford-components. In [10], Harbater realises all finite groups G as quotients of $\text{Gal}(\overline{\mathbb{Q}_p}(T)/\mathbb{Q}_p(T))$ by pasting cyclic covers

$$\mathbb{P}^1 \rightarrow \mathbb{P}^1, z \mapsto z^n$$

in a clever way. He thus gets global orbifold charts $\varphi: X \xrightarrow{/G} \mathcal{S}$ for rational orbifolds $\mathcal{S} = (\mathbb{P}^1, (\zeta_1, e_1), \dots, (\zeta_{2r}, e_{2r}))$ with $e_i = e_{i+r}$ for $i = 1, \dots, r$ and a Mumford curve Y . He does this with ‘mock’ covers of $\mathbb{P}_{\mathbb{F}_p}^1$ lifting to totally degenerated curves ramified above $2r$ points of the projective line, if the group G is generated by r elements. The translation of Harbater’s proof in [6, Section 6.4] to $*$ -trees shows that the marked points have to be separable into pairs ζ_i, ζ_{i+r} by a pure affinoid covering of \mathbb{P}^1 . This means that from each vertex in the $*$ -tree for the chart φ either two cusps emanate or none at all. In this special case, all vertex stabilisers are cyclic, and all edge groups are trivial.

The ramification data $\mathbf{C} = (C_1, \dots, C_{2r})$ for those covers with conjugation classes of the deck group G are required to have a representative of the form

$$(g_1, \dots, g_r, g_1^{-1}, \dots, g_r^{-1}).$$

This motivates M. Fried to call it an *Harbater-Mumford-representative* of \mathbf{C} .

In [9, 3.21] M. Fried proves that the following condition on G and \mathbf{C} with a Harbater-Mumford-representative forces the subspace $\mathcal{H}_0(G, \mathbf{C})$ of the Hurwitz space $\mathcal{H}_0(G, 2r)$ with ramification type \mathbf{C} to be a component (we shall call it a *Harbater-Mumford-component*):

HM If any pair of inverse conjugation classes is removed from \mathbf{C} , then the remaining $2r - 2$ classes generate the group G .

Proposition 3.1. *Let G be any finite group. Then Harbater-Mumford-components of the Hurwitz space $\mathbf{H}_0(G, 2r)$ contain covers of rational Mumford orbifolds by Mumford curves if and only if G is a quotient of a free tree product of cyclic groups.*

Proof. If a Harbater-Mumford component contains covers of Mumford orbifolds by Mumford curves, then G is a quotient of the fundamental group of a $*$ -tree with trivial edge groups, as we have seen in the beginning of this subsection.

Let $G = \Pi/\Sigma$ be a quotient of a free tree product Π of cyclic groups. The group Π is the fundamental group of a finite tree with cyclic vertex groups and trivial edge groups which can be made into a discontinuously embeddable tree by subdividing segments of the type as in Figure 1.

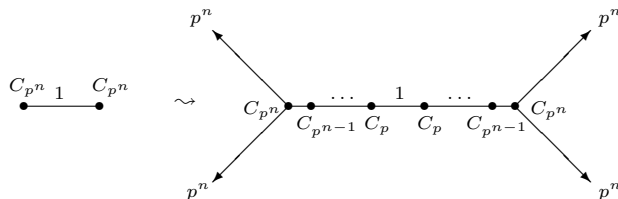


FIGURE 1. Subdivision of a segment gives a discontinuously embeddable tree with isomorphic fundamental group.

The non-trivial edge groups are the intersections of the stabilisers of the edges' extremities. The tree on the right of Figure 1 is realizable because it is the connected sum of two copies of the $*$ -tree for C_{p^n} with an allowed segment: Indeed, the segment with the trivial edge group belongs to Herrlich's list [12] of segments of groups embeddable into the Bruhat-Tits tree. The geodesics left and right are the $*$ -trees of C_{p^n} , and the paths joining the segment to the other two trees are nothing but a part of the fixed point "strip" of a p -group in the Bruhat-Tits tree.

More generally, making explicit what Kato proposes in [16, 8.1], a segment whose fundamental group is the free product $C_e * C_{e'}$ with $e = mp^r$, $e' = m'p^{r'}$ and $\gcd(m, p) = \gcd(m', p) = 1$ can be made realisable by subdividing in the way as in Figure 2.

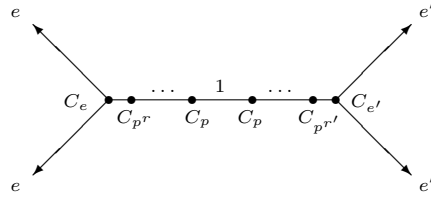


FIGURE 2. A discontinuous embedding of an amalgam of cyclic groups.

The reason is that the two decomposition groups contain p -groups with thick fixed point strips.

All other segments with vertex groups of order prime to p are realisable [12]. Thus, we have made the tree into a $*$ -tree whose fundamental group embeds discontinuously into $\mathrm{PGL}_2(\mathbb{C}_p)$. Let $\Omega \subseteq \mathbb{P}^1$ be its domain of regularity. It is also the domain of regularity for the normal subgroup Σ of finite index. Then we have a commuting diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{\Sigma} & X \\ & \searrow \Pi & \downarrow G \\ & & \mathbb{P}^1 \end{array}$$

where X is a Mumford curve. □

3.2. $*$ -trees with trivial edge groups. First, we consider $*$ -trees T having only trivial edge groups. As the Bass-Serre fundamental group $\pi_1^{\mathrm{BS}}(T)$ of a tree of groups is generated by the vertex groups, this is also the case for all quotients of $\pi_1^{\mathrm{BS}}(T)$. In particular, our Galois group G is generated by generators of the vertex groups, which we interpret as local monodromies in the following way:

T is obtained by pasting together local data: elementary $*$ -trees are glued by connecting them with edges at appropriate vertices. Let now T' be a stable tree of groups obtained from T by contracting edges whose stabiliser equals one of its extremities' groups. Now, by Belyi's Theorem the Mumford-Schwarz-orbifolds corresponding to those elementary $*$ -trees are defined over $\bar{\mathbb{Q}}$. This gives us local (complex) monodromies γ_0, γ_∞ resp. $\gamma_0, \gamma_1, \gamma_\infty$ around the cusps $0, \infty$ resp. $0, 1, \infty$, obeying the relations

$$\gamma_0 \gamma_1 = 1 \quad \text{resp.} \quad \gamma_0 \gamma_1 \gamma_\infty = 1.$$

This motivates the following

Definition 3.2. Let $\mathbf{C} = (C_1, \dots, C_r)$ be a generating system of conjugation classes of G having the property that there exist representatives $\gamma_i \in C_i$ and a partition of the set $\{1, \dots, r\}$ into pairs and triples in such a way that for each pair i, j resp. each triple i, j, k the relation

$$\gamma_i \gamma_j = 1 \quad \text{resp.} \quad \gamma_i \gamma_j \gamma_k = 1$$

holds and all $\langle \gamma_i, \gamma_j \gamma_k \rangle$ are isomorphic to subgroups of $\mathrm{PGL}_2(\mathbb{C}_p)$. This ramification datum \mathbf{C} is called of Mumford type.

We may now formulate a special case of a Riemann Existence Theorem for Mumford curves:

Theorem 2 (RET of Mumford type). *Let G be a finite group. Then the Mumford-Hurwitz space $\mathfrak{H}_0(G, \mathbf{C})$ is non-empty if \mathbf{C} is of Mumford type.*

Proof. Let \mathbf{C} be of Mumford type. Taking any connected sum of $*$ -trees for the groups generated by the chosen pairs and triples of $\gamma_i \in C_i$ as described above, we get a tree T of groups which can be made discontinuously embeddable by subdividing similarly as in the proof of Proposition 3.1. If we denote by N the fundamental group of T , then G is obviously a quotient of N , and the Galois chart $\varphi: X \rightarrow \mathfrak{S}$ is constructed in an analogous way as the HM-cover from the proof of Proposition 3.1. \square

3.3. Virtual ramification in $*$ -trees. The motivation for this subsection is the example [1, III.6.4.6] of a Mumford-Schwarz-orbifold \mathfrak{S} whose orbifold fundamental group $\pi_1^{\mathrm{orb}}(\mathfrak{S}, \bar{s})$ is not topologically generated by its local monodromies γ_0, γ_1 around the points $0, 1$. Neither does there exist a local monodromy γ_∞ around ∞ such that $\gamma_0, \gamma_1, \gamma_\infty$ obey the relation $\gamma_0 \gamma_1 \gamma_\infty = 1$. The reason for this is that already a quotient of $\pi_1^{\mathrm{orb}}(\mathfrak{S}, \bar{s})$ isomorphic to a fundamental group of a $*$ -tree with three cusps, does not allow this. A discrete quotient of $\pi_1^{\mathrm{orb}}(\mathfrak{S}, \bar{s})$ is called a *p-triangle group (of Mumford type)*.

In [17] there is a first attempt towards classifying all discrete p -adic triangle groups. Refining the methods used there yields a complete classification of all discrete subgroups of $\mathrm{PGL}_2(K)$ for finite extensions K of \mathbb{Q}_p [7]. In what follows we will make use of this classification result.

Proposition 3.3. *Every indecomposable discrete subgroup N of $\mathrm{PGL}_2(K)$ generated by elements of finite order is an amalgamation which corresponds to a tree of groups $\mathbf{T}(N)$ with the following properties:*

- (1) *The vertex groups N_v are finite non-cyclic groups or p -adic triangle groups.*
- (2) *The edges e correspond to maximal cyclic subgroups of $N_{o(e)}$ resp. $N_{t(e)}$.*
- (3) *For each branch point x of N_v there is at most one edge e with $o(e) = v$, and N_e is the stabiliser of x .*

Sketch of proof. (Cf. [7]). Suppose, that there is an indecomposable discrete subgroup N of $\mathrm{PGL}_2(K)$ not satisfying the properties of the proposition. It is a finite amalgam of m_N maximal finite subgroups. Choose N such that m_N is minimal under all counterexamples.

We claim that no pair of finite groups occurring in the amalgam is amalgamated along a cyclic group unless one of the finite groups is itself cyclic.

One sees that there are no two maximal finite subgroups in this amalgam which are amalgamated along a common maximal cyclic subgroup, otherwise one would easily obtain a counterexample N' with $m_{N'} < m_N$, a contradiction.

It remains the case that two maximal finite non-cyclic subgroups G_1, G_2 in the amalgam are amalgamated along a common cyclic subgroup G_0 which is not maximal in one of them. One can then see that $G_1 *_G G_2$ is an amalgam of a finite group and a

triangle group along a common maximal cyclic subgroup. Again it follows that m_N is not minimal, a contradiction.

As a consequence, N is isomorphic to an amalgam of finite non-cyclic groups along non-cyclic subgroups amalgamated also with cyclic groups along cyclic groups. Such a group is necessarily a triangle group. \square

In [7] there is also a complete classification of all discrete p -adic triangle groups.

Definition 3.4. *The amalgam and the tree of groups from Proposition 3.3 are called regular.*

Corollary 3.5. *The number of branch points for an indecomposable discrete subgroup N of $\mathrm{PGL}_2(K)$ is the number of vertices in $\mathbf{T}(N)$ plus two.*

Proof. Note that all vertices of the regular tree give a contribution of three branch points. The statement now follows easily from Theorem 1 together with the fact that N is the fundamental group of a tree of groups. \square

Proposition 3.6 (F. Kato, F. Herrlich). *There are no infinite discrete p -adic triangle groups, if $p > 5$. For $p \leq 5$ there are infinitely many.*

Proof. The first statement is a result of the classification in [7]. The second statement follows from the classification of discontinuously embeddable segments of finite groups [12]. \square

Definition 3.7. *A system of conjugation classes $\mathbf{C} = (C_0, C_1, C_\infty)$ of a finite group G is called of Mumford-Schwarz type, if there is a G -cover $X \xrightarrow{G} \mathbb{P}^1$ with ramification data \mathbf{C} and with a Mumford curve X .*

REGULAR AMALGAMS

Let \mathbf{T} be a regular tree of groups for a regular amalgam Γ . For any subtree of groups $T_0 \subseteq \mathbf{T}$ denote $c_{\mathbf{T}}(T_0)$ the number branch points of the cover $\varpi_{\Gamma}: \Omega \rightarrow \mathbb{P}^1$ which are contributed by the subtree T_0 .

Lemma 3.8. *Let a regular tree \mathbf{T} be spanned by disjoint subtrees T_1 and T_2 through an edge e . Then*

$$c_{\mathbf{T}}(T_1) = c_{T'}(T_1)$$

for any subtree T' of \mathbf{T} containing T_1 and e .

Proof. As regular amalgams are of type **Am**, Corollary 2.14 says that the set of branch points corresponding to T' is the pushout of the branch loci of T_1 and $T_2 \cap T'$ along the two branch points corresponding to e . Hence, the contribution from T_1 does not depend on the complementary tree $T_2 \cap T'$. \square

Lemma 3.9. *Let \mathbf{T} be a regular tree of groups, and v a vertex of \mathbf{T} with non-cyclic stabiliser N_v . Then*

$$c_{\mathbf{T}}(v) = 3 - \mathrm{ord}(v).$$

In particular, the order of any vertex in \mathbf{T} is at most three.

Proof. Let $\mathrm{ord}(v) = m$, which is a non-negative number. The vertex v is connected by m edges to some components T_1, \dots, T_m . We have

$$c_{\mathbf{T}}(\mathbf{T}) = c_{\mathbf{T}}(v) + \sum_{i=1}^m c_{\mathbf{T}}(T_i) = c_v(v) - 2m + \sum_{i=1}^m c_{T_i}(T_i).$$

The equation on the right holds true because of Theorem 1. Now, if T^i is the tree spanned by T_i and v , then

$$c_{T_i}(T_i) = c_{T^i}(T_i) + 1,$$

because

$$c_{T^i}(T_i) + c_{T^i}(v) = c_{T^i}(T^i) = c_{T_i}(T_i) - 2 + c_v(v) = c_{T_i}(T_i) + 1,$$

and $c_{T^i}(v) = 2$. The latter can be seen by first using Lemma 3.8 in order to see that $c_{T^i}(v) = c_{[w,v]}(v)$, where w is the vertex in \mathbf{T} nearest to v , and then verifying $c_{[w,v]}(v) = 2$ which follows from the fact that the geodesic in the Bruhat-Tits tree between the two fixed points of the edge stabiliser gets folded under the action of N_v . Since, by Lemma 3.8, $c_{T^i}(T_i) = c_{\mathbf{T}}(T_i)$, we have in fact

$$c_{\mathbf{T}}(\mathbf{T}) = 3 - m + \sum_{i=1}^m c_{\mathbf{T}}(T_i),$$

from which we conclude that

$$c_{\mathbf{T}}(v) = 3 - m.$$

This number is non-negative, if and only if $m \leq 3$. \square

Definition 3.10. A Galois cover $f: X \xrightarrow{G} \mathbb{P}^1$ with a Mumford curve X is called regular, if the fundamental group N of its $*$ -tree is indecomposable. In this case the induced cover $\Omega \xrightarrow{N} \mathbb{P}^1$ factoring over f is also called regular.

Theorem 3 (Virtual RET). Let G be a finite group and $\mathbf{C} = (C_1, \dots, C_r)$ some conjugation classes in G with $\gamma_i \in C_i$. Then there is a Galois cover $X \xrightarrow{G} \mathbb{P}^1$ with a Mumford curve X and ramification type $\mathbf{C} = (C_1, \dots, C_r)$, if and only if there is a partition

$$\{1, \dots, r\} = \bigcup_j I_j$$

into sets I_j of cardinality at least two such that for each $I := I_j$ the tuple $C_I = (C_i)_{i \in I}$ gives rise to a regular cover $\Omega_I \xrightarrow{\Gamma_I} \mathbb{P}^1$.

Proof. The proof of Theorem 2 carries over to this more general situation. \square

The structure of regular trees now yields:

Theorem 4. If $X \xrightarrow{G} \mathbb{P}^1$ is a non-cyclic regular cover with ramification type $\mathbf{C} = (C_1, \dots, C_r)$ and $\gamma_i \in C_i$, then there is a system $\mathbf{g} = (\gamma_1, \dots, \gamma_r, \gamma_{r+1}, \dots, \gamma_{3s})$ of elements in G and a partition of $\{1, \dots, 3s\}$ into triples $I = \{i_0, i_1, i_\infty\}$ such that

$$C_I = (C(\gamma_{i_0}), C(\gamma_{i_1}), C(\gamma_{i_\infty}))$$

is of Mumford-Schwarz type for some p -adic triangle group $\Delta_I \xrightarrow{\varphi_I} G$ mapping to G . The conjugation classes in C_I are to be taken in the image of φ_I .

Proof. Let the regular cover be given as

$$\begin{array}{ccc} \Omega & \xrightarrow{F} & X \\ & \searrow N & \downarrow G \\ & & \mathbb{P}^1 \end{array}$$

with F a finitely generated free group, and let \mathbf{T} be the corresponding regular tree. For each vertex v of \mathbf{T} we have a p -adic triangle group Δ_v and $c_{\mathbf{T}}(v)$ cusps with generators $\gamma \in G$ of their stabilisers. This locally defines a cover $\varphi_v: X_v \xrightarrow{\Delta_v} \mathbb{P}^1$ with $3 - c_{\mathbf{T}}(v)$

more branch points. Setting $F_v := F \cap \Delta_v$, a finitely generated free group, we obtain a commuting diagram with exact rows

$$\begin{array}{ccccccc} 1 & \longrightarrow & F_v & \longrightarrow & \Delta_v & \longrightarrow & G_v \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & F & \longrightarrow & N & \longrightarrow & G \longrightarrow 1 \end{array}$$

As the left and the middle vertical arrows are injective, so is also the right vertical map. Hence, the decomposition groups of φ_v embed into G , and we can complete the $c_{\mathbf{T}}(v)$ conjugacy classes of chosen elements of G with further elements of G to a system of Mumford-Schwarz type for the local cover φ_v . Doing this for all vertices of \mathbf{T} yields s Mumford-Schwarz type systems of conjugacy classes in G or, in other words, the required $\mathbf{g} = (g_1, \dots, g_{3s})$ containing all the γ_i ($i = 1, \dots, r$) together with a partition of $\{1, \dots, r\}$ into triples as in the theorem. \square

Definition 3.11. *The system of conjugacy classes \mathbf{C} of G as in Theorem 4 is called virtually of Mumford type.*

The reason for calling Theorem 3 virtual RET lies in the structure of regular trees, which conceal local monodromies in their vertex groups. This can be seen most easily when there are no triangle groups present.

Corollary 3.12. *If, in the situation of Theorem 4, all triangle groups Δ_I are finite, then the system \mathbf{g} generates G and each triple $I = \{i_0, i_1, i_\infty\}$ ($I = I_j$ from a partition as in Theorem 4) can be chosen such that the relation*

$$\prod_{i \in I} \gamma_i = 1$$

holds for some order on I .

Proof. Let N be the fundamental group of the tree of groups associated with the regular cover. As all the triangle groups Δ_I are finite, the relations

$$\prod_{i \in I} \delta_i = 1$$

are feasible with elements $\delta_i \in \Delta_I$, hence in G . As N is generated by all vertex groups of \mathbf{T} , it follows that the set of all $\gamma_i = \varphi_I(\delta_i)$ generates G . \square

Definition 3.13. *The γ_i from Corollary 3.12 are called virtual monodromies.*

Remark 3.14. As remarked by Y. André in [1, Remark III.6.4.7], infinite p -adic triangle groups of Mumford type are not generated by their local monodromies (i.e. generators of the decomposition groups of the branch points), there exist finite quotients with the same property. This is why Corollary 3.12 does not hold in the presence of infinite vertex groups. In the general case that \mathbf{T} has infinite triangle groups as vertex groups, an exact description of virtual local monodromies as generating systems of Galois groups is yet to be found.

3.4. From trees to graphs. In order to generalise the preceding to covers $\varphi: X \xrightarrow{G} \mathcal{S}$ with an arbitrary Mumford orbifold \mathcal{S} , it is convenient to use the following setup: let \mathfrak{G} be the $*$ -graph corresponding to the global chart φ of \mathcal{S} , where we assume X to be a Mumford curve of genus g . Further, let $N := \pi_1^{\text{BS}}(\mathfrak{G})$ be the Bass-Serre fundamental group of the graph of groups \mathfrak{G} . Assuming that N is of type **Am**, there is a commuting

diagram with exact rows and surjective vertical arrows

$$(2) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \langle N_{\bullet} \rangle_N & \longrightarrow & N & \longrightarrow & F_g \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \langle N_{\bullet} \rangle_G & \longrightarrow & G & \longrightarrow & F \longrightarrow 1 \end{array}$$

where $\langle N_{\bullet} \rangle_N$ is the normal subgroup of N generated by the vertex and edge groups of \mathfrak{G} . It is isomorphic to the fundamental group of a $*$ -tree obtained by deleting g appropriate edges of \mathfrak{G} . One sees readily that the top row of (2) splits.

Definition 3.15. *If for a Mumford curve X of genus g , the group N in the diagram*

$$(3) \quad \begin{array}{ccc} \Omega & \xrightarrow{F_g} & X \\ & \searrow N & \downarrow G \\ & & Y \end{array}$$

is of type **Am**, then we say that the Galois cover $X \xrightarrow{G} Y$ is also of type **Am**.

Here is now a first generalised virtual Riemann Existence Theorem:

Theorem 5. *The Mumford-Hurwitz space $\mathfrak{H}_g(G, \mathbf{C})$ contains a cover of type **Am**, if and only if G contains a normal subgroup H containing representatives of conjugacy classes from \mathbf{C} such that \mathbf{C} , viewed as a ramification type for H , is virtually of Mumford type, G/H is a quotient of F_g and the exact sequence*

$$1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1$$

splits.

Proof. \Rightarrow . Let a cover as in (3) be given with N of type **Am**. Then we obtain a commutative diagram with exact rows and columns

$$(4) \quad \begin{array}{ccccccc} & & 1 & & 1 & & 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & F_b & \longrightarrow & F_h & \longrightarrow & F_c \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \langle N_{\bullet} \rangle_N & \longrightarrow & N & \longrightarrow & F_g \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & F \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 1 & & 1 & & 1 \end{array}$$

whose lowest row is the second row of (2). We still need to show that the lowest row of (4) splits. As the middle row splits, the images of $N_0 := \langle N_{\bullet} \rangle_N$ and F_g under the middle vertical arrow to G intersect trivially. So, one sees that there is a map $F \rightarrow G$ making the lowest row split.

\Leftarrow . We construct the cover of type **Am** by showing that there is a diagram (4). For this, we need to construct a group N of type **Am** and a free group F_g fitting into the diagram. As the cover of \mathbb{P}^1 corresponding to H is virtually of Mumford type, the left column is exact with a free group F_b on b generators and with middle group $N_0 = \langle N_{0\bullet} \rangle_{N_0}$. By taking a $*$ -tree for N_0 , and making its finite part into a graph

of genus g by inserting g edges with trivial stabilisers, one obtains a group N of type **Am** fitting into the center of (4). Thus, obviously, the middle row is split. Further, the vertical map from F_g is the quotient map from $G \rightarrow F$, which implies that the vertical map from N is surjective, as the map $N_0 \rightarrow H$ is also surjective. It remains to show that F_h is free, finitely generated. As F_c is free in c generators, mapping these generators to F_h in such a way that the top right square commutes, is possible with a map $\alpha: F_c \rightarrow F_h$. Since the two low rows are split, it follows that α is injective. Thus, F_h is generated by two free groups which intersect trivially, hence free. \square

If, however, the G -cover $\varphi: X \rightarrow Y$ is not of type **Am**, then let N' be an amalgamification of N . The inclusion $N' \subseteq N$ extends to a commutative diagram with exact rows

$$(5) \quad \begin{array}{ccccccccc} 1 & \longrightarrow & F_{g'} & \longrightarrow & N' & \longrightarrow & G' & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & F_g & \longrightarrow & N & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

where $F_{g'} = N' \cap F_g$ is free. Thus, the Galois group G' of the cover of type **Am** can be viewed as a subgroup of G . By the construction of amalgamification, G' contains the decomposition groups of φ and therefore the corresponding cover φ' contains the ramification datum of φ . Since the corresponding graph of groups \mathfrak{G} and \mathfrak{G}' are obtained from one another by a finite sequence of steps (Am 1) and (Am 2) as in the proof of Lemma 2.12, we can describe, how to obtain G from G' . Namely, if \mathfrak{G}' has s loops affected by (Am 1) and t pairs of non-cyclic isomorphic vertex groups $N'_{i_1} \cong N'_{i_2}$ of two different vertices v_{i_1}, v_{i_2} obtained by (Am 2), then

$$(6) \quad G = \langle G', \alpha_1, \dots, \alpha_s, \gamma_{e_1}, \dots, \gamma_{e_t} \rangle$$

subject to relations

$$(7) \quad \gamma_{e_i} N'_{i_1} \gamma_{e_i}^{-1} = N'_{i_2}, \quad i = 1, \dots, t$$

(and possibly other relations), where the γ_{e_i} are images of edges e_i affected by the amalgamification.

Proposition 3.16. *If the Mumford-Hurwitz space $\mathfrak{H}_g(G, \mathbf{C})$, $\mathbf{C} = (C_1, \dots, C_r)$, is non-empty, then G contains a subgroup G' containing an element γ_i of each C_i such that $\mathfrak{H}_{g'}(G', \mathbf{C}')$ contains a cover of type **Am** with $*$ -graph \mathfrak{G}' (where $\mathbf{C}' = (C_{G'}(\beta_i))_{i=1, \dots, r+2s+3t}$ are the conjugacy classes of γ_i and of conjugates of α_j in G'), and such that G is the quotient of an HNN-extension N of $N' = \pi_1^{\text{BS}}(\mathfrak{G}')$ where (6) and (7) hold with $s+t \leq g$.*

Proof. It is clear by the discussion above, that amalgamification gives a cover of type **Am** with the asserted properties. \square

Let now a G -cover φ be given which allows for a subgroup G' a cover φ' of type **Am** such that (6), (7) and $s+t \leq g$ hold true. It is easy to see N making the diagram (5) commute with exact rows and injective vertical arrows. By assertion, N is a HNN-extension of N' having N' as an amalgamification. But often N is not discretely embeddable into $\text{PGL}_2(K)$.

4. REALISATION OF GROUPS

4.1. Mumford curves of genus ≥ 2 . Harbater's result follows easily from Proposition 3.1 and can be improved slightly:

Theorem 6. *Every finite group G is the Galois group of a cover $X \xrightarrow{G} \mathbb{P}^1$ with a Mumford curve X of genus $g \geq 2$.*

Proof. The group G , generated by cyclic subgroups G_1, \dots, G_r , is a quotient of the free product $G_1 * \dots * G_r$. This solves, by Proposition 3.1, the original inverse Galois problem. Since, as seen in the proof of Proposition 3.1, we may realize every cyclic group C_n as a covering group of a tree with four cusps whose stabilisers are all C_n , the Riemann-Hurwitz-formula tells us that the genus of the corresponding C_n -cover is at least 2, if $n > 2$. C_2 -covers are realized as hyperelliptic covers in a similar way: 6 cusps are sufficient for genus ≥ 2 . \square

The actual cover can be constructed as in [10]. Translated into the language of $*$ -trees, the proof reads:

Let G be generated by two already realised subgroups H_1 and H_2 (with Mumford curves of genus ≥ 2). This means that there are rational Mumford orbifolds $\mathcal{S}_1 = (\mathbb{P}^1, (\zeta_{11}, e_{11}), \dots, (\zeta_{1q}, e_{1q}))$ and $\mathcal{S}_2 = (\mathbb{P}^1, (\zeta_{2,q+1}, e_{2,q+1}), \dots, (\zeta_{2r}, e_{2r}))$, global charts $\varphi_1: X_1 \xrightarrow{H_1} \mathcal{S}_1$, $\varphi_2: X_2 \xrightarrow{H_2} \mathcal{S}_2$ with Mumford curves X_1, X_2 . Let T_i be the $*$ -tree for φ_i . The cover φ_i induces a morphism of graphs of groups such that the following diagram is Cartesian:

$$\begin{array}{ccc} X_i & \xrightarrow{H_i} & \mathcal{S}_i \\ \downarrow & & \downarrow \\ \Gamma_i & \xrightarrow{H_i} & T_i \end{array}$$

where the vertical arrows are induced by Tate reduction maps.

Let $T_i^+ = T_i \# v$ be obtained from T_i by adjoining to any vertex of T_i an edge e (with trivial group) terminating in v (not belonging to T_i). Γ_i^+ is then defined by the Cartesian square

$$\begin{array}{ccc} \Gamma_i^+ & \longrightarrow & \Gamma_i \\ H_i \downarrow & & \downarrow H_i \\ T_i^+ & \xrightarrow{\text{contr.}} & T_i \end{array}$$

Finally, let

$$\Delta_i^+ := G/H_i \times \Gamma_i^+, \quad \Delta_i := G/H_i \times \Gamma_i.$$

Now, $\Delta_i^+ \setminus \Delta_i = G \times e$ for $i = 1, 2$, and we can paste $\Delta_i^+ \rightarrow T_i^+$ along the morphisms $(\Delta_i^+ \setminus \Delta_i \rightarrow T_i^+ \setminus T_i) \cong (G \times e \xrightarrow{\text{triv.}} e)$, obtaining

$$\tilde{\varphi}: \Delta \xrightarrow{/G} T = T_1^+ \#_e T_2^+$$

where $T_1^+ \#_e T_2^+$ means the tree obtained by pasting the trees T_i^+ along the edge e . We see that Δ is connected and T a $*$ -tree for an r -punctured orbifold

$$\mathcal{S} = (\mathbb{P}^1, (\zeta_{ij}, e_{ij}), i = 1, 2, j = 1, \dots, r)$$

obtained by lifting $\tilde{\varphi}$ to a chart fitting into

$$\begin{array}{ccc} X & \xrightarrow{G} & \mathcal{S} \\ \downarrow & & \downarrow \\ \Delta & \xrightarrow{G} & T \end{array}$$

with a Mumford curve X and deck group G .

The results of Section 3.4 lead to the natural generalisation of Theorem 6 stated in [20, Corollary 1.2] without explicit proof¹:

Theorem 7. *For every Mumford curve S of genus g , every finite group G is the Galois group of a cover $X \xrightarrow{G} S$ with a Mumford curve X .*

Proof. Realise G as a cover of the projective line by a Mumford curve by giving an explicit $*$ -tree T^* with fundamental group N . Let T be the finite part of T^* . Inserting g edges between any vertices of the finite part gives a discontinuously embeddable graph of groups with fundamental group² N_g , thus realising G over a Mumford curve S' by exhibiting a point of $\mathfrak{M}(N_g)$. By forgetting the cover and the n branch points, one sees that there is a map

$$\mathfrak{M}(N_g) \rightarrow \mathfrak{M}_g$$

between the $(3g-3+n)$ -dimensional moduli space $\mathfrak{M}(N_g)$ and the $(3g-3)$ -dimensional space \mathfrak{M}_g of Mumford curves of genus g . This map is easily seen to be surjective. \square

4.2. The full automorphism group. The extra care taken for realising groups as deck groups in higher genus allows us to prove

Theorem 8. *Every finite group is isomorphic to the full automorphism group of a Mumford curve.*

Proof. Let G be the finite group. Consider the Mumford-Hurwitz space $\mathfrak{H}_g^h(G, \underline{e})$ of covers $X \xrightarrow{G} Y$ with signature $\mathbf{e} = (e_1, \dots, e_n)$ with Mumford curves X and Y of genera h and g , respectively, where h and g are related by the Riemann-Hurwitz formula (the notational redundancy is used here for convenience). We have a natural map

$$\Phi: \mathfrak{H}_g^h(G, \mathbf{e}) \rightarrow \mathcal{M}_h,$$

into the moduli space of all smooth curves of genus h , whose image $\mathcal{M}_h(G, \mathbf{e})$ is of pure dimension $3g-3+n$. We take a component \mathcal{M} of $\mathcal{M}_h(G, \underline{e})$. According to [19, Theorem 5.1], if

$$(*) \quad (g, n) \notin \{(2, 0), (1, 2), (1, 1), (0, 4), (0, 3)\},$$

then $\mathcal{M}^\circ := \{X \in \mathcal{M} \mid \text{Aut } X = G\}$ is open and dense in \mathcal{M} . It follows that \mathcal{M}° intersects $\mathfrak{M}_h(G, \mathbf{e}) := \Phi(\mathfrak{H}_g^h(G, \mathbf{e}))$, if and only if \mathcal{M} does, since $\mathfrak{M}_h(G, \mathbf{e})$ is of the same pure dimension $3g-3+n$, provided $\mathfrak{H}_g^h(G, \mathbf{e}) \neq \emptyset$ (Lemma 2.6). If the Mumford-Hurwitz space is non-empty, there exist components \mathcal{M} which intersect the Mumford locus $\mathfrak{M}_h(G, \mathbf{e})$, however. So, we need only to find for each finite group G a non-empty Mumford-Hurwitz space $\mathfrak{H}_g^h(G, \mathbf{e})$ fulfilling $(*)$. But this is possible by Theorem 7: just take $g = 3$. \square

Let us give an alternative proof using only the realisation over \mathbb{P}^1 : Theorem 6 allows us to realise G as the deck group of a cover $\varphi: X \rightarrow \mathbb{P}^1$ with a Mumford curve X of genus $g \geq 2$. If G is of order greater than 5, Harbater's solution to the inverse Galois problem gives more than 4 branch points, and we are done. Otherwise, take the $*$ -tree of φ and add two edges in such a way that the resulting graph is of genus two. As it is also a $*$ -graph, there is a corresponding Galois cover $X' \rightarrow S'$ fulfilling $(*)$, and we have realised G as $\text{Aut } X'$ with a Mumford curve X' (of genus ≥ 2).

¹which has been added during the time of refereeing the original version of this article [21, Theorem 1.2]

²of type **Am**

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