Mumford dendrograms and discrete p-adic symmetries [1]

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Abstract

In this article, we present an effective encoding of dendrograms by embedding them into the Bruhat-Tits trees associated to p-adic number fields. As an application, we show how strings over a finite alphabet can be encoded in cyclotomic extensions of $\mathbb{Q}_p$ and discuss p-adic DNA encoding. The application leads to fast p-adic agglomerative hierarchic algorithms similar to the ones recently used e.g. by A. Khrennikov and others. From the viewpoint of p-adic geometry, to encode a dendrogram X in a p-adic field K means to fix a set S of K-rational punctures on the p-adic projective line $\mathbb{P}^1$. To $\mathbb{P}^1 \setminus S$ is associated in a natural way a subtree inside the Bruhat-Tits tree which recovers X, a method first used by F. Kato in 1999 in the classification of discrete subgroups of $\mathrm{PGL}_2(K)$.

Next, we show how the p-adic moduli space $\mathfrak{M}_{0,n}$ of $\mathbb{P}^1$ with n punctures can be applied to the study of time series of dendrograms and those symmetries arising from hyperbolic actions on $\mathbb{P}^1$. In this way, we can associate to certain classes of dynamical systems a Mumford curve, i.e. a p-adic algebraic curve with totally degenerate reduction modulo p.

Finally, we indicate some of our results in the study of general discrete actions on $\mathbb{P}^1$, and their relation to p-adic Hurwitz spaces.

1 Introduction

Mumford curves arise as the generalisation of the so-called *Tate uniformisation* of p-adic elliptic curves [13, §6]. The latter has a combinatorial description as a $\mathbb{Z}$-action on the real line "connecting" the points 0 and ∞ over $\mathbb{Q}_p$. The crucial idea by Mumford [10] was to view the real line as a geodesic line inside the Bruhat-Tits tree $\mathscr{T}_{\mathbb{Q}_p}$ for $\mathrm{PGL}_2(\mathbb{Q}_p)$ and to consider a discrete action of a subgroup G generated by g hyperbolic fractional linear transformations acting regularly on a subdomain Ω of the p-adic Riemann sphere $\mathbb{P}^1$. It turns out that the orbit space $X = \Omega/G$ is a complete algebraic curve of genus g, and that not all p-adic algebraic curves admit such a uniformisation. A curve of the form $X = \Omega/G$ as above is called a *Mumford curve* or a *p-adic Riemann surface*.

Here, we are concerned with the application of p-adic geometry in the analysis of hierarchical data. From a geometric viewpoint, the tree $\mathscr{T}_{\mathbb{Q}_p}$ represents the hierarchical

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organisation of all p-adic numbers, including ∞. Namely, a p-adic number can in a natural way be viewed as an infinite path inside the tree $\mathscr{T}$ starting from some vertex v. Two paths starting from v correspond to two p-adic numbers having their first terms coincide in their p-adic expansions. The more terms they have in common, the closer they are p-adically. Hence, for some given p-adic numbers, their geodesic paths in $\mathscr{T}_{\mathbb{Q}_p}$ will yield a subtree which hierarchically represents their proximities. This motivates the usefulness of the Bruhat-Tits tree for hierarchical data analysis by finding a way of encoding data as p-adic numbers. Unfortunately, there is no natural way of doing this for arbitrary data other than strings over an alphabet.

Time series of hierarchical data naturally yield the consideration of families of sets of p-adic numbers. The corresponding geometric construct is a *moduli space* of such families. Here, they come in the form of $M_{0,n}$, the p-adic *moduli space of n-pointed genus zero curves*. Classically, these and their variants in higher genus play an important role in string theory, and we expect this also to be the case in p-adic string theory. However, for data mining, a time series is simply a sequence of points in $M_{0,n}$, and it would certainly be interesting to be able to interpolate and have a curve inside the moduli space in order to say something about the evolution of the time series, or the probability of a certain behaviour in time.

2 Dendrograms

Dendrograms are a certain way of depicting trees arising in the hierarchical classification of data. Their intention is usually to describe hierarchies found within some given dataset. However, it is often the result of imposing hierarchies onto the data, depending on the choice of a metric. A lot of work by Fionn Murtagh aims to find ultrametricity in data in order to reveal underlying hierarchy, e.g. [11, 12]. The reason is precisely the tree-like structure of any ultrametric distance. From a p-adic viewpoint, the following procedure seems natural:

1. Encode dataset $X = \{x_1, \dots, x_n\}$ by p-adic numbers Y.

2. Construct the dendrogram for X from the code Y.

The dendrogram for X is uniquely determinded by Y and can be computed quite fast. Hence, the true problem is to find a suitable encoding by p-adic numbers. This is in general a very difficult task, as one is likely to need the dendrogram *a priori*. However, for strings of letters from a given alphabet, we will show how p-adic encodings can be effected in Section 5.

A more precise definition of a dendrogram is that of a metrised tree with finitely many ends, all of which are labelled.

In what follows, we assume that to each dataset X, there is given a dendrogram D which is supposed to reveal the hierarchical structure within X.

3 p-adic dendrograms

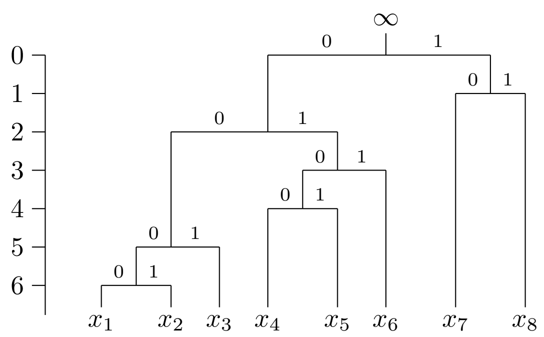


Figure 1: A 2-adic dendrogram.

Consider the dendrogram D as depicted in Figure 1. If one goes down from ∞ along a path in D to some datum $x = x_i$, and picks up the labels 0 or 1 along the way, then one gets a 2-adic encoding

$$x = \sum_{\text{level } \nu} a_\nu 2^\nu \in \mathbb{Q}_2,$$

where coefficient a_ν is the number picked up at level ν. Here, this yields the numbers

$$\begin{array}{llll} x_1 = 0, & x_2 = 2^6, & x_3 = 2^5, & x_4 = 2^2, \\ x_5 = 2^2 + 2^4, & x_6 = 2^2 + 2^3, & x_7 = 1, & x_8 = 2^0 + 2^1 \end{array}$$

Of course, the procedure yields just finite 2-adic expansions of rational numbers. Notice that any permutation of data labels x_i yields the 2-adic code in different order. This is equivalent to permuting branches in D, which leads to different representations of the dendrogram in the Euclidean plane.

In any case, the whole dendrogram D gets embedded into an infinite tree: the *Bruhat-Tits tree* $\mathscr{T}_{\mathbb{Q}_2}$ for the group $\mathrm{PGL}_2(\mathbb{Q}_2)$. For a general prime number p, the tree $\mathscr{T}_{\mathbb{Q}_p}$ is a locally finite $p+1$-regular tree. The latter means that from each vertex there are precisely $p+1$ emanating edges. The reason is that the vertices can be interpreted as p-adic discs, and the edges are given by maximal non-trivial inclusion of discs. It is known that each disc has precisely p maximally smaller subdiscs and lies inside precisely one minimal bigger disc. Hence, each vertex has precisely p children vertices and one parent vertex. This is illustrated in Figure 2.

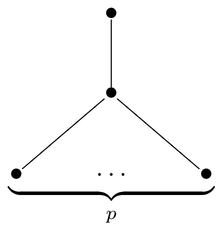


Figure 2: Local structure of $\mathscr{T}_{\mathbb{Q}_p}$.

The number p of children vertices comes from the isomorphism $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p$, saying that the residue field of $\mathbb{Q}_p$ is the finite field $\mathbb{F}_p \cong \{0, 1, \dots, p-1\}$ with p elements. Hence, each downward edge can be labelled by any representative for $\mathbb{F}_p$ in the ring $\mathbb{Z}_p$ of p-adic integers. Quite common is e.g. the set of labels $\{0, \dots, p-1\}$.

Moving downwards from some vertex v on will end in some p-adic number $x \in \mathbb{Q}_p$ as the intersection of a decreasing sequence of discs corresponding to the vertices on the infinite path, and picking up labels as before yields a p-adic expansion of x as a Laurent series in p. Hence, all of $\mathbb{Q}_p$ can be considered as lying at the boundary of $\mathscr{T}_{\mathbb{Q}_p}$. However, there is one more boundary point of $\mathscr{T}_{\mathbb{Q}_p}$ outside $\mathbb{Q}_p$: taking the path going upwards from each vertex will lead to the point ∞. Hence, we have found

$$\partial \mathscr{T}_{\mathbb{Q}_p} = \mathbb{Q}_p \cup \{\infty\} = \mathbb{P}^1(\mathbb{Q}_p),$$

where the latter space $\mathbb{P}^1$ is the p-adic *projective line*.

We have seen even more that the local picture of the Bruhat-Tits tree allows a local interpretation as another projective line: namely, there is a bijection

$$\{\text{edges emanating from vertex } v\} \cong \mathbb{F}_p \cup \{\infty\} = \mathbb{P}^1(\mathbb{F}_p),$$

with the projective line defined this time over the residue field $\mathbb{F}_p$.

Let now be given a finite set $X = \{x_0, \dots, x_n\}$ of p-adic numbers. Then by taking inside the Bruhat-Tits tree $\mathscr{T}_{\mathbb{Q}_p}$ all geodesic paths between the points of X, one obtains a subtree $\mathscr{T}(X)$ which we call a *p-adic dendrogram*. We give credit to Fumiharu Kato who used this construction already in 1999 in the classification of p-adic discrete projective linear groups (cf. also [8, §5.2]).

Observe that the 2-adic encoding of a dendrogram described above yields a 2-adic dendrogram for the 2-adically coded data plus the extra "datum" ∞. This extra point at infinity allows to determine the root of a dendrogram, from which all paths to genuine data are oriented downwards, i.e. passing through children vertices. In Figure 1, the root corresponds to the unit disc, because on the one hand the data code contains $0, 1 \in \mathbb{Q}_2$, and the three numbers $0, 1, \infty$ uniquely determine the p-adic unit disc as the intersection of the three geodesics in $\mathscr{T}_{\mathbb{Q}_p}$ connecting $0, 1, \infty$. And on the other hand, all data are encoded by numbers within the p-adic unit disc

$$\mathbb{D} = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$$

which is characterised by the fact that the p-adic expansion of its elements contains no negative powers of p.

4 Non-binary data

In the previous section, we have seen how to p-adically encode data having a *binary* dendrogram, and we have defined p-adic dendrograms which are not necessarily binary. Hence, a natural way of encoding data whose dendrogram D is not binary would be by increasing the prime number p to the size of at least the maximal number of children vertices in D.

However, there is an alternative way of doing this without changing the prime p. Namely, consider any finite field extension K of $\mathbb{Q}_p$. It is well-known that the p-adic norm extends uniquely to a norm $|\ |_K$, and that K is complete for this norm. Again, the unit disc is the ring $O_K = \{x \in K \mid |x|_K \leq 1\}$, and the next smaller disc containing 0 is πO_K, where π is a so-called *uniformiser* and plays the role of the prime p in K. It holds true that the residue field

$$\kappa := O_K / \pi O_K \cong \mathbb{F}_q$$

is a finite field extension of $\mathbb{F}_p$ with $q = p^f$ elements for some natural number $f \geq 1$.

In general, it holds true that

$$f = \dim_{\mathbb{F}_p}(\kappa) \leq \dim_{\mathbb{Q}_p}(K) =: n, \tag{1}$$

where the dimensions are of that of vector spaces over the scalar fields $\mathbb{Q}_p$ and $\mathbb{F}_p$, respectively. The result is that there are more discs defined over K than over $\mathbb{Q}_p$. More precisely, the number of "children" disks has increased to $q = p^f$, and there is a new Bruhat-Tits tree $\mathscr{T}_K$ which is again infinite, but this time $q + 1$-regular. The analogue holds true:

$$\partial \mathscr{T}_K \cong \mathbb{P}^1(K)$$

$$\{\text{edges emanating from vertex } v\} \cong \mathbb{P}^1(\kappa).$$

In fact, there is an embeding of trees

$$\mathscr{T}_{\mathbb{Q}_p} \to \mathscr{T}_K$$

given in general by subdividing edges and increasing the number of edges emanating from a vertex. Note that that the subdivision of edges comes from a relation between the uniformisers:

$$|\pi|_K^e = |p|_p,$$

which causes the length of an edge in $\mathscr{T}_K$ to be an e-th fraction of an edge length in $\mathscr{T}_{\mathbb{Q}_p}$. The number e is called the *ramification index* of the field extension $K/\mathbb{Q}_p$. By adopting the labelling method from above, we obtain the general encoding

$$x = \sum_{\nu=-m}^{\infty} a_\nu \pi^\nu,$$

where a_ν is taken from a system $\mathcal{R}$ of representatives in K for the residue field κ. This is nothing but the π-adic expansion of elements from K.

In the case that (1) is an equality, the field extension $K/\mathbb{Q}_p$ is called *unramified*. By the well known formula

$$n = e \cdot f,$$

this is equivalent to $e = 1$. In this case, the prime p can be taken as the uniformiser of K, and we obtain again p-adic expansions—only with more choice of coefficients. A special case is given by a so-called *cyclotomic* extension $K = \mathbb{Q}_p(\zeta)$ obtained by adjoining to $\mathbb{Q}_p$ the powers of a primitive $(p^f - 1)$-th root ζ of unity. This case is known to be unramified, and we can take as coefficients

$$\mathcal{R}_f := \left\{0, \zeta, \dots, \zeta^{p^f-2}\right\} \tag{2}$$

for the p-adic expansion of elements from K. Note, that for $f = 1$, this yields a set of coefficients different from the usual choice $\{0, \dots, p-1\}$.

The proofs for most of the statements in this section can be found in [7, Ch. 5].

5 Strings over an alphabet

Let $\mathcal{A}$ be a finite alphabet. We will show how to realise p-adic encodings of strings over $\mathcal{A}$.

First, denote by $S(\mathcal{A})$ the set of all strings with letters from $\mathcal{A}$. The subset of finite strings will be denoted by $S_{\text{fin}}(\mathcal{A})$. Now, for f sufficiently large, any injective map

$$\mathcal{A} \to \mathcal{R}_f,$$

with $\mathcal{R}_f$ defined as in (2), induces an encoding of $S(\mathcal{A})$ in O_K, where K is the cyclotomic field $\mathbb{Q}(\zeta)$ with ζ a primitive $(p^f - 1)$-th root of unity. Clearly, the finite strings are then in bijection with the set of polynomials $\mathcal{R}_f[p]$ in the prime p whose coefficients are from $\mathcal{R}_f$. We even have more [2, Thm. 3.1]:

Theorem 5.1 *There exists a cyclotomic field $K = \mathbb{Q}_p(\zeta)$ with ζ as above, and a closed isometric embedding $\phi\colon S(\mathcal{A}) \to O_K$ such that $\phi(S_{\text{fin}}(\mathcal{A})) \subseteq \mathcal{R}_f[p]$, and is dense in $\phi(S(\mathcal{A}))$.*

Here, the metric on $S(\mathcal{A})$ is given by the *Baire distance*

$$\delta_p(x, y) := \inf \left\{ p^{-n} \mid \text{first } n \text{ letters of } x \text{ and } y \text{ coincide} \right\}$$

Note that the image $\phi(S(\mathcal{A}))$ is a disc with holes coming from the complement of $\phi(\mathcal{A})$ in $\mathcal{R}_f$. More precisely, if $x \in O_K$ is represented as

$$x = \sum_{\nu \in \mathbb{N}} a_\nu p^\nu, \ a_\nu \in \mathcal{R}_f,$$

the holes are given as the union of open discs

$$\{a_0 \notin \phi(\mathcal{A})\} \cup \{a_1 \notin \phi(\mathcal{A})\} \cup \{a_2 \notin \phi(\mathcal{A})\} \cup \dots$$

Note further, that although there are only finitely many encodings $\phi\colon \mathcal{A} \to \mathcal{R}_f$, there are infinitely many p-adic encodings by changing the system $\mathcal{R} \subseteq O_K$ of representatives for the residue field κ.

6 p-adic clustering

If data X are encoded p-adically, it is a very simple and fast task to retrieve the uniquely determined hierarchical structure of D given by the tree $\mathscr{T}(X)$. Any clustering algorithm using the p-adic metric will never need to change the metric when measuring distances between disjoint clusters C_1 and C_2, because of the fact

$$\text{dist}_p(C_1, C_2) = |x - y|_p$$

for any $x \in C_1$, $y \in C_2$. Essentially, the fact that one seeks a subtree of a tree makes things more simple and faster than in the archimedean situation. In [2, §3], an explicit form of a p-adic hierarchic classification algorithm has been discussed. Benois-Pineau et al. have applied such an algorithm in image segmentation [1].

7 DNA

As an example for what has been said in the previous sections, we discuss p-adic encoding of DNA. Here, the alphabet is given as $\mathcal{A} = \{A, G, C, T\}$, where

$$A = \text{Adenine} \qquad G = \text{Guanine}$$
$$C = \text{Cytosine} \qquad T = \text{Thymine}$$

Dragovich and Dragovich [6] choose a 5-adic encoding in the field $\mathbb{Q}_5$

$$\phi_{\mathrm{DD}} \colon \mathcal{A} \to \mathcal{R} = \{0, 1, 2, 3, 4\}$$

with $\phi_{\mathrm{DD}}(\mathcal{A}) = \{1, 2, 3, 4\}$. This allows for taking 0 as a "blank" in order to separate words made out of $\mathcal{A}$. So, in fact, they use the extended alphabet $\mathcal{A} \cup \{\text{"blank"}\}$ and encode it with a bijection to $\mathcal{R}$ taking the "blank" to 0.

Khrennikov and Kozyrev [9] use a bijection

$$\mathcal{A} \to \mathbb{F}_2^2$$

as their encoding. As an $\mathbb{F}_2$-vector space, $\mathbb{F}_2^2$ is isomorphic to the additive group of the finite field $\mathbb{F}_{2^2}$ with four elements. This field, in turn, is the residue field of the cyclotomic field $K = \mathbb{Q}_2(\zeta)$ with ζ a primitive third root of unity. Because of the correspondence

$$1 \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \zeta \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad 1 + \zeta \leftrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

their encoding can be interpreted as choosing $\mathcal{R}_{\mathrm{XK}} = \{0, 1, \zeta, 1 + \zeta\}$ and a bijection

$$\phi_{\mathrm{XK}} \colon \mathcal{A} \to \mathcal{R}_{\mathrm{XK}},$$

which gives a 2-adic encoding. However, there is no "blank" in this case.

By the previous sections, we see that there are a lot more possibilities, even for 2-adic encodings. For a version without "blank", a bijection with

$$\mathcal{R}_2 = \left\{0, 1, \zeta, \zeta^2\right\}$$

could be used. And for a version with "blank", an injection into

$$\mathcal{R}_3 = \left\{0, 1, \xi, \dots, \xi^6\right\}$$

could be interesting, where ξ is a seventh root of unity[2].

The following questions come up naturally:

Questions 7.1 *1. Are there among the possible* 2*-adic encodings*

$$\mathcal{A} \to \mathcal{R}_3$$

some more preferred than others from the point of view of genomics?

2. Which are the best choices for systems $\mathcal{R} \subseteq O_K$ of representatives for the residue field $\kappa = \mathbb{F}_{2^3}$ from a genomic point of view (possibly including "blank")?

Of course, there is the question, whether cyclotomic or unramified p-adic fields are sufficiently suited for genomics.

[2]Notice that $7 = 2^f - 1$ with $f = 3$.

8 Time series

Assume that we are given some time dependent p-adically encoded data, i.e. a set of p-adic numbers

$$S_t = \{s_0(t), \dots, s_n(t)\}$$

at some instances of time $t = 0, 1, \dots, N$. We assume that $\infty \in S_t$ and that there are no "collisions" at any time, if we may use the language of "particles" moving inside some "space". This corresponds to the p-adic projective line with $n + 1$ points removed:

$$X_t := \mathbb{P}^1 \setminus S_t,$$

which is the usual way of denoting an $n + 1$-*pointed genus zero curve*. If we normalise for each t via some fractional linear map

$$z \mapsto \frac{az + b}{cz + d}$$

the punctures S_t to contain $0, 1, \infty$, we have in X_t a standard representative of a point x_t inside the *moduli space* $M_{0,n+1}$ *of* $n+1$*-pointed genus zero curves* defined over $\mathbb{Q}_p$. In the language of moduli spaces, the time series S_t corresponds to a family X_t of punctured curves which in turn comes from a map

$$\{0, 1, \dots, N\} \to M_{0,n+1}.$$

Collisions can also be treated in this way: simply replace $M_{0,n+1}$ by a suitable compactification $\bar{M}_{0,n+1}$ in which the boundary corresponds to all possible ways of colliding particles.

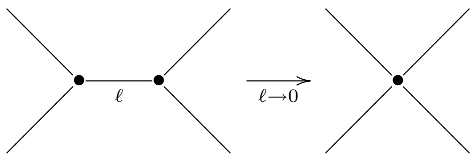


Figure 3: Edge contraction.

There is now an infinite-to-one map

$$\Pi \colon M_{0,n+1} \to \mathcal{D}_n, \; \mathbb{P}^1 \setminus S \mapsto T(S)$$

into the space $\mathcal{D}_n$ of all dendrograms for n data and ∞. The fibre of a point $x \in \mathcal{D}_n$ corresponds to the infinitely many possible p-adic encodings of the dendrogram associated to x. Hence, these correspond to the *sections* $f\colon \mathcal{D}_n \to M_{0,n}$, i.e. maps satisfying

$$\Pi \circ f = \mathrm{id}_{\mathcal{D}_n}.$$

The space $\mathcal{D}_n$ is a polyhedral complex of dimension

$$\dim \mathcal{D}_n = \dim M_{0,n+1} = (n+1) - 3,$$

where the subtraction of 3 comes from the normalisation after which 3 points are fixed. The maximal cells $\mathcal{D}_n$ are all of the dimension of the moduli space and consist of the binary dendrograms. A cell in $\mathcal{D}_n$ is characterised by the fact that the abstract trees corresponding to its elements are all isomorphic, whereas the edge lengths vary. Passing to a neighbouring cell amounts to contracting an edge as illustrated in Figure 3.

This geometric approach allows for considering the time series S_t as a sequence of points on some (unknown) path inside $M_{0,n}$. The analyst would then e.g. want to know which paths are more likely than others in order to understand the dynamical behaviour of the system and to estimate dendrograms at times outside the instances $t = 0, 1, \ldots, N$.

9 Genus 1 time series

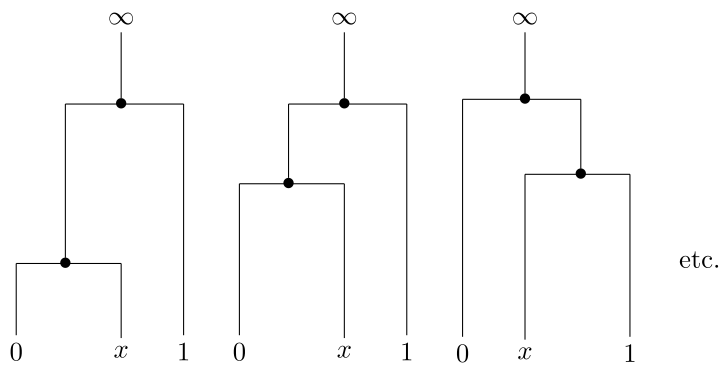


Figure 4: A sequence of dendrograms.

Consider the sequence of dendrograms as given in Figure 4. We can view this as a vertex v_t determined by x at time t "jumping" along the geodesic line between 0 and 1 with respect to the fixed vertex determined by ∞, as in Figure 5.

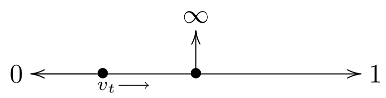


Figure 5: Vertex jump.

If the vertex jumps at a constant rate, i.e. the distance at each time is the same, then we can model this by a translation along that geodesic line, or p-adically via a Möbius transformation

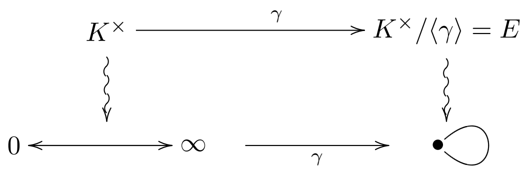
$$\gamma\colon z \mapsto \frac{-1}{(1-c)z-1} \in \mathrm{PGL}_2(K),$$

where $|c| < 1$. This corresponds for v_t to a jump of distance $-\log_p|c|$ to the right. This is the case of γ being hyperbolic.

By a change of coordinates taking $(0, 1, \infty)$ to $(0, \infty, 1)$ we transform everything said above to a hyperbolic action of the cyclic group $\langle\gamma\rangle$ on the geodesic line between 0 and ∞, or p-adically: on $K^\times = \mathbb{P}^1 \setminus \{0, \infty\}$. Hence, γ has now the form

$$\gamma\colon z \mapsto c \cdot z,$$

and we obtain the commutative diagram



in which E is a so-called *Tate elliptic curve*. It is a p-adic curve of genus 1, and the vertical wiggly arrows are the so-called *reduction* or *tropicalisation maps*.

The above example generalises to the case of a discrete action on the p-adic projective line $\mathbb{P}^1$ of a group G of fractional linear transformations inside $\mathrm{PGL}_2(K)$. If, in this case $\Omega \subseteq \mathbb{P}^1$ is the domain on which Γ acts without limit or fixed points, then $C = \Omega/\Gamma$ is known to be a so-called *Mumford curve*, a p-adic analogon of *Riemann surface*.

10 Identifying p-adic Riemann surfaces

Mumford curves are considered for p-adic higher genus string amplitudes in [5], where the authors call them *p-adic Riemann surfaces*. Unlike in the classical case, not all algebraic curves defined over $\mathbb{Q}_p$ are p-adic Riemann surfaces. However, Chekhov et al. conjecture that the other curves do not contribute to the p-adic (or adelic) string amplitude [5, Conjecture §4.3]. This brings another physical motivation to the general problem of recognising Mumford curves among algebraic curves.

Notice that the loop in the commutative diagram of the preceding section is of length $-\log_p|c|$ and hence shrinks to zero, if $|c|$ approaches unity. In this case, the fractional linear transformation γ is not hyperbolic, and there is no longer a discrete action of $\langle\gamma\rangle$ on the geodesic.

On the side of elliptic curves, this corresponds to the fact that the family of elliptic curves parametrised by γ converges to a p-adic elliptic curve which is not a Tate curve. Such curves do exist, and they can be distinguished by their j-invariant.

In fact, let the elliptic curve E be given by an equation over K in Legendre normal form

$$E\colon y^2 = x(x-1)(x-\lambda),$$

where we may assume that $|\lambda| = 1$ (this implies also $|\lambda - 1| \le 1$). Then, if K is a sufficiently large finite extension of $\mathbb{Q}_p$, it holds true by [4, Ex. 3.8] that

$$\begin{aligned} E \text{ is a Tate curve} &\Leftrightarrow |j(E)| > |2|_p^4 \\ &\Leftrightarrow |\lambda - 1| < |2|_p^2 \end{aligned}$$

This result was already known for the case $p > 2$, in which $|2|_p = 1$ [13, Thm. 5]. The last equivalence follows from a well-known formula relating λ and the j-invariant. In order to show that the first and third statements are equivalent, one can consider the cover $\phi\colon E \to \mathbb{P}^1$ of degree 2 defined by the Legendre equation: ϕ is simply projection onto the x-coordinate. This induces a cover of degree 2 of the tree $\mathscr{T}(\{0, 1, \infty, \lambda\})$ as depicted in Figure 6. The proof then consists of calculating the infimum of ℓ for which the upper graph still represents a Tate curve [4]. For $p = 2$, the intuition $\inf(\ell) = 0$ fails because of too many fixed vertices of the elliptic involution on the Bruhat-Tits tree.

In the case of general Mumford curves (or p-adic Riemann surfaces, if one wishes), one can study the cover $\Omega \to \Omega/G = C$ in a similar combinatorial way. Here, the so-called *Hurwitz spaces*, which are moduli spaces for covers between curves, come into play. It turns out that the question whether the upper curve in a cover $f\colon X \to Y$ is a Mumford curve is subtle. Only a restricted type of covers f can in principle allow X to be a Mumford curve, and even then the answer depends on the position of the branch points of the covering map f [3, 4].

11 Conclusion

A p-adic encoding of hierarchical data has been discussed from a geometric point of view. Any dendrogram can in this way be viewed as a subtree of the Bruhat-Tits tree for

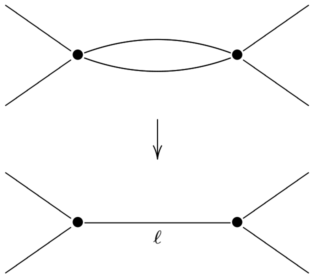


Figure 6: Tate curve covering $\mathbb{P}^1$, graphically.

$\mathrm{PGL}_2(K)$ defined over a p-adic field K large enough to encapture the maximal number of children vertices in the dendrogram. This is possible without changing the prime number p. The philosophical result is that cluster analysis becomes the finding of a suitable p-adic encoding of data, because then the dendrogram is uniquely determined by the ultrametric geometry. As an example, strings over a finite alphabet have been considered, where the p-adic distance coincides with the Baire distance. Application to encoding of DNA has been discussed, where the general question is raised which arithmetic conditions on a 2-adic field K must be imposed from the point of view of genomics.

A consideration of time series of hierarchical data leads to families of dendrograms or n-pointed p-adic projective lines and their moduli spaces as a natural geometric framework. Higher genus p-adic algebraic curves come into the scene, if a time series can be modelled via a discrete action of fractional linear transformations on the p-adic Riemann sphere. This and p-adic multiloop calculations in string theory [5] motivate the question of how to decide whether a given algebraic curve of higher genus is a p-adic Riemann surface.

It is the hope that methods from p-adic string theory and enumerative geometry will eventually find their way into hierarchical data analysis.

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