RESEARCH REPORT

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2. Title of Research in Japan. Orbifold uniformisation of p-adic surfaces

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4. Research background. Mustafin [4] defines the Bruhat-Tits building and shortly describes a completion without proofs. A detailed account thereof from the viewpoint of algebraic groups is found in [6]. Harm Voskuil's thesis [5] systematically treats nonarchimedean Hopf surfaces: such are uniformised by infinite discrete subgroups of $\mathrm{PGL}_3(K)$ which are not free. The viewpoints in the cited references are mostly of a geometric nature.

The particular aim of the research at Kyoto university was to lay the foundation for a systematic combinatorial study of the uniformisation of surfaces by discrete subgroups of $\mathrm{PGL}_3(K)$ over a p-adic field K, modelled on the successful one-dimensional case.

5. Research methodology. For our purposes, a more elementary and combinatorial approach than in [6] was pursued. Therefore, a combinatorial study of the geometry of the Bruhat-Tits building Δ for $\mathrm{PGL}_3(K)$, as well as its boundary, was conducted from scratch. Special attention was given to the action of $F_{21} \cong C_7 \rtimes C_3$ on Δ, a finite group not occurring in the one-dimensional case.

6. Research implementation and results. Starting from an apartment $\mathfrak{A} = \mathfrak{A}(P, Q, R)$ in Δ for three points $P, Q, R \in \mathbb{P}^2_K$ in general position, it is known that the choice of a vertex in $\mathfrak{A}$, yields a coordinate system on $\mathfrak{A}$ which identifies every vertex of $\mathfrak{A}$ with a unique point in $\mathbb{N}^3$. This allows to define the notion of *halfline* in $\mathfrak{A}$, a notion which in fact does not depend on the choice of a coordinate system. We say that a halfline is *K-rational*, if it is a path in $\mathfrak{A}$, i.e. if subsequent vertices in the halfline are connected by edges in $\mathfrak{A}$. Now, generalising the concept of ends from the one-dimensional case, we define a *ray* to be an equivalence class of paths γ in Δ for which there exists an apartment $\mathfrak{A}$ containing all but finitely many vertices of γ and such that $\gamma \cap \mathfrak{A}$ is a halfline. We mean paths to be equivalent, if they differ only in finitely many vertices.

Definition. *The* Werner-Mustafin boundary *of Δ is the set of all K-rational rays in Δ.*

Parallelness of rays is well-defined by taking representing halflines in apartments. A key observation is that there is a one-to-one correspondence between the rays in $\mathfrak{A}$ which are parallel to the halfline ρ_{PQ} given by $\{[\mathcal{O}e_P \oplus \mathcal{O}e_Q \oplus \pi^n \mathcal{O}e_R] \mid n \in \mathbb{N}\}$ and the vertices of the geodesic line $]P, Q[$ in the Bruhat-Tits tree for the line $PQ \cong \mathbb{P}^1$. A similar observation holds for P, Q replaced by any other two distinct points from P, Q, R. Here, the vectors $e_P, e_Q, e_R \in K^3$ represent $P, Q, R \in \mathbb{P}^2$ in general position[1]. In this way we were able to consider the boundary of Δ as given by the compactified one-dimensional Bruhat-Tits buildings of the K-rational lines in $\mathbb{P}^2$.

Sectorial subdivision and resolution of singularities

As the Bruhat-Tits building, viewed as a combinatorial object associated to the special fibre of $\mathbb{P}^2_K$, remains unchanged after birational changes of the generic fibre, we gave apartments a further structure allowing to keep track of performed blowups. This is important for dealing with the resolution of singularities.

Consider an apartment $\mathfrak{A} = \mathfrak{A}(P, Q, R)$ for three points $P, Q, R \in \mathbb{P}^2$ in general position, and a fixed vertex v of $\mathfrak{A}$. Any two of the three halflines $\rho_{P'Q'}$, where $P', Q' \in \{P, Q, R\}$ are distinct, determine a sector in $\mathfrak{A}$ at v of angle $120°$, called a *natural sector*. Any halfline from v within the interior of Σ corresponds to a sequence of lattices converging to the same point in $\mathbb{P}^2$. This fact does not depend on the choice of the vertex v. The natural sectors form the *natural sectorial subdivision* of the apartment $\mathfrak{A}$ at v. Any additional halflines from v in $\mathfrak{A}$ yield a *sectorial subdivision* of $\mathfrak{A}$ or of sectors in $\mathfrak{A}$ defined by previously given halflines from v. We interpret a halfline h ocurring in any sectorial subdivision of $(\mathfrak{A}, v)$ as exceptional divisor in a blowup of $\mathbb{P}^2$ in $L \cap L'$, where $\Sigma_v(L, L')$ is the sector containing h.

Simplicial torsion and rationality

By a *simplicial torsion* we mean an automorphism of Δ leaving invariant some simplex σ in Δ without fixing any of its vertices. It was found that any element $\gamma \in \mathrm{PGL}_3(K)$ of finite order having three K-rational invariant lines in $\mathbb{P}^2$ cannot be a simplicial torsion. The idea for proving this is that $\langle\gamma\rangle$ would contain an element ω of order 2 or 3 having an invariant apartment $\mathfrak{A}$ containing σ. But ω would then not fix any vertex of $\mathfrak{A}$, a fact which can be lead into a contradiction. As a by-product, Herrlich's calculation of the fixed locus of finite transformations in dimension one [2, Lemma 3] was generalised.

A building for F_{21}

By Blichfeldt's list of finite groups in the two-dimensional case [1], the group $F_{21} \cong C_7 \rtimes C_3$ has the matrix representation

$$S = \begin{pmatrix} \zeta & & \\ & \zeta^2 & \\ & & \zeta^4 \end{pmatrix}, \quad T = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix},$$

where ζ is a primitive seventh root of unity. The generator S has precisely one invariant apartment $\mathfrak{A}(S) = \mathfrak{A}(P, Q, R)$, where $P = (1 : 0 : 0)$, $Q = (0 : 1 : 0)$ and $R = (0 : 0 : 1)$. $\mathfrak{A}(S)$ is also invariant under the second generator T, containing $v = [\mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \mathcal{O}e_3]$ as the only fixed vertex, where e_1, e_2, e_3 is the standard basis of K^3. The smallest interesting F_{21}-invariant subbuilding of Δ is spanned by $\mathfrak{A}(S)$ and the orbits under $\langle S\rangle$ of the apartment $\mathfrak{A}(T)$ defined by the three fixed points in $\mathbb{P}^2$ of T. One calculates that $\mathfrak{A}(S) \cap \mathfrak{A}(T)$ is either empty ($p = 3$) or a vertex ($p \neq 3$). $\mathfrak{A}(S)$ encaptures the singularity induced by the C_3-action and trisecting the natural sectors corresponds to the two blowups resolving the singularity.

[1] Note that changing those representing vectors results in a parallel halfline

The C_7-singularities, however, remain undetected. So, the influence of adding more apartments will have to be checked. E.g. $\bigcup_{Q' \in PQ,\, R' \in PR} \mathfrak{A}(P, Q', R')$ or finite invariant apartment subsystems should be considered. This means calculating $\mathfrak{A}(S) \cap \mathfrak{A}(P, Q', R')$, the stabilised part under C_7.

Some geometry of apartments

The observation that 2 distinct K-rational points on $\mathbb{P}^2$ and a vertex $v \in \Delta$ determine a unique sector of 120° in some apartment containing v lead us to prove that four points $A, B, C, D \in \mathbb{P}^2$ in general position determine a unique vertex $v(A, B, C, D)$ as the intersection of the four apartments given by the points. Intersecting only two of the apartments yields a sector of 120°.

If the fourth point D is on the line through two of the first three, then the intersection is a sector of 180°. The position of its bounding line ℓ within the apartment depends on the distance of D to A or B, if $D \in AB$. The ray given by one half of ℓ corresponds to the vertex in the Bruhat-Tits tree for $AB \cong \mathbb{P}^1$ determined by A, B, D.

Five points in general position yield two apartments $\mathfrak{A}(A, B, C)$, $\mathfrak{A}(A, D, E)$ intersecting in a K-rational halfline which becomes a line if instead E is placed on BC. If also $D \in BC$, then the intersection is a strip of possibly positive width equal to the size of the one-dimensional intersection $]B, C[\cap]D, E[$, and if D is on AB or AC, then it is a sector of 60°. The remaining case with $D \in AB$ and $E \in AC$ gives a sector of 120°.

The relation of $\mathfrak{A}(A, B, C)$ and $\mathfrak{A}(D, E, F)$ for six points in $\mathbb{P}^2$ is combinatorially a bit more involved. In the generic case, the six special vertices $v(A, B, C, D)$ etc. will play an essential role. As an intermediary step, the distance of a vertex $v \in \Delta$ to an apartment $\mathfrak{A}$ can be expressed in terms of crossratios from the one-dimensional theory with the help of such special vertices using F. Kato's result [3, 3.5.1].

7. International exchange achieved through the research.

8. Major publications of research results.

References

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